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Four Precision Points Geometric Design of Spatial 3R Manipulators

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Abstract: In this paper, the four precision points geometric design problem of serial-link robot manipulators with three revolute (R) joints is solved using a polynomial homotopy continuation method. At each precision point, the position and orientation are defined. The dimensions of the geometric parameters of the 3-R manipulator are computed so that the manipulator will be able to place its end-effector at these four pre-specified locations. Denavit and Hartenberg parameters and 4x4 homogeneous matrices are used to formulate the problem and obtain twenty-four design equations in twenty-seven design unknowns. Three of the design parameters are set as *free choices* and their values are selected arbitrarily. Two different cases for selecting the free choices are considered and their design equations are solved using polynomial homotopy continuation. In both cases for free choice selection, thirty-six distinct manipulators are found that will be able to place their end-effector at the four specified spatial positions and orientations.

Keywords: Geometric Design, Robot Manipulators, Polynomial Homotopy Continuation

1 Introduction

The calculation of the geometric parameters of a multi-articulated mechanical or robotic system so that it guides a rigid body in a number of specified spatial locations or *precision points* is known as the *Rigid Body Guidance Problem*. It is also called the *Geometric Design Problem* [1]. The precision points are described by six parameters: three for position and three for orientation. This problem has been studied extensively for planar mechanisms and robotic systems and has recently drawn much attention to researchers for spatial multi-articulated systems. Solution techniques for the geometric design problem may be classified into two categories: *exact synthesis* and *approximate synthesis*.

Exact synthesis methods result in manipulators which guide a rigid body exactly through the specified precision points. Solutions in the exact synthesis exist only if the number of independent design equations obtained by the precision points is less than or equal to the number of design parameters. If the number of design equations is less than the number of design parameters, then several of the design parameters can be regarded as free choices and their values can be selected arbitrarily so that a well-determined system is obtained. The number of precision points that may be prescribed for a given mechanism or manipulator is limited by the system type and the number of design parameters that are selected to be free choices [2]. This number can be calculated using Tsai and Roth's formula [3], [4].

In *approximate synthesis*, using an optimization algorithm, a mechanism is found that, although not guiding a rigid body exactly through the desired poses, it optimizes an objective function defined using information from all the desired poses. Approximate synthesis is mainly used in *over-determined* geometric design problems where more precision points are defined than allowed for exact synthesis and therefore no exact solution exists. A complete listing of the extensive amount of research that has been performed in the geometric design of spatial mechanisms and robotic systems, both exact and approximate, can be found in [5].

The equations for the geometric design problem of mechanisms and manipulators are mathematically represented by a set of non-linear, highly coupled multivariate polynomial equations. The solutions of these equations can be obtained by either numerical continuation methods or algebraic methods [6].

Using algebraic methods, the exact synthesis of planar mechanisms for rigid body guidance has been studied extensively by many researchers and is described in most textbooks on mechanism synthesis [7], [8]. The exact synthesis of a few spatial mechanisms and manipulators has been solved using algebraic methods. The spatial geometric design problems that captured the most attention were the spatial revolute-revolute (R-R) [9]-[12] and the cylindrical-cylindrical (C-C) manipulators [13]-[15]. Other than these two dyads, the geometric design problem has been solved algebraically for the following spatial manipulators/mechanisms. Innocenti [16] solved the geometric design problem for the sphere-sphere binary link. Neilsen and Roth [15] solved the slider-slider sphere dyad, cylinder-cylinder binary link, revolute-slider-sphere dyad and cylinder-sphere binary link design problem. McCarthy [17] also solved the exact synthesis problem for several types of dyads. Recently, Lee and Mavroidis [18] had solved the prismatic-revolute-revolute (PRR) design problem using elimination technique. Even though algebraic methods had been demonstrated to be very effective in solving several geometric design problems for spatial mechanical systems, there exist many types of

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robotic and mechanical systems that are used frequently, such as the 3R, 4R and 5R manipulators, for which the exact synthesis of the geometric design problem has not been solved before. The main reason for this is the high complexity of the non-linear polynomial design equations that are obtained.

Polynomial continuation methods have been used extensively in the geometric analysis and design of mechanisms and robotic systems [19]. They have been very efficient in solving very difficult problems in the geometric design and analysis of robot manipulators for which no algebraic solution had been discovered. The most representative examples are the continuation solutions of the inverse kinematic problem of the general 6R serial manipulator [20] and the forward kinematic problem of the general parallel platforms [21]. For both problems continuation methods calculated correctly all solutions and they served as targets for the algebraic solutions that followed soon after. Roth and Freudenstein [22] were the first to use continuation methods to solve polynomial systems obtained in the kinematic synthesis of mechanisms. Morgan and Wampler [23] solved the path following design problem of 4R closed loop, planar mechanisms using continuation method when five precision points and the location of the fixed pivots are defined. The same problem was solved with continuation method by Wampler, Morgan and Sommese [24] when nine precision points are specified and no free choices are made. These researchers showed that there are up to 4,326 distinct design solutions for this problem. Dhingra, Cheng and Kohli [25] solved several design problems for six link, slider crank and four-link planar mechanisms using polynomial continuation methods. Recently, Lee and Mavroidis [1] applied polynomial continuation method in spatial mechanisms synthesis and solved the spatial 3R three precision points geometric design problem.

This paper extends the work done by Lee and Mavroidis to solve the spatial 3R four precision points geometric design problem. For this problem, Tsai [3] and Roth [26], used screw theory, were the first to obtain the design equations. However, they did not solve them. Lee and Mavroidis [1] solved the three precision points design problem with two different types of free choices selection and found eight distinct solutions for each. Lee, Mavroidis and Merlet [27] had studied the 3R five precision points design problem using interval analysis and found all the real solutions in a predefined bounded domain.

In the problem studied in this paper, four spatial positions and orientations are defined and the dimensions of the geometric parameters of the 3-R manipulator are computed so that the manipulator will be able to place its end-effector at these four pre-specified locations. Denavit and Hartenberg (DH) parameters and 4x4 homogeneous matrices are used to formulate the problem and obtain twenty-four design equations in twenty-seven design unknowns. Three of the design parameters are set as *free choices* and their values are selected arbitrarily. Two different cases for selecting the free choices are considered and their design equations are solved using polynomial continuation. In both cases for free choice selection, thirty-six distinct manipulators are found that will be able to place their end-effector at the four specified spatial positions and orientations. The polynomial homotopy continuation method is implemented using the software package PHC developed by Verschelde and Cools [28], [29].

2 Polynomial Continuation

Polynomial continuation is a numerical method that computes all the solutions of a system of polynomial equations by tracing a finite number of solution paths from a polynomial start system to the target system of interest. There are two main steps in using polynomial continuation method: a) generate the start system, b) trace a finite number of solution paths to obtain all the solutions of the polynomial system of interest.

A start system is always generated based on an upper bound on the number of solutions of the target system. There are several known upper bounds, the only one relevant to us is the multi-homogeneous Bezout bound [6]. The computation of this bound is based on a partition of the variables. Suppose that we have a system of n equations in n unknowns $x_1, \cdots, x_n$. We can partition the n variables into m groups using the following way:

$$G_1 = \{x_{11}, \cdots, x_{1k_1}\},\ G_2 = \{x_{21}, \cdots, x_{2k_2}\},\ \ldots,$$
$$G_m = \{x_{m1}, \cdots, x_{mk_m}\} \text{ with } \bigcup_{i=1}^{m} G_i = \{x_1, \cdots, x_n\} \qquad (1)$$

The subscript k_i is the number of variables in the i-th group. The maximum degree of the j-th equation, with respect to the i-th variable group will be denoted by the parameter d_{ij}. The multi-homogeneous Bezout bound (also known as the m-homogeneous number) with respect to this partition is defined as the coefficient of the product $\prod_{i=1}^{m} \alpha_i^{k_i}$ in the polynomial $\prod_{j=1}^{n}\left(\sum_{i=1}^{m} d_{ij}\alpha_i\right)$ where $\alpha_1, \cdots, \alpha_m$ are indeterminants introduced for bookkeeping. In general, different partition will result in different multi-homogeneous Bezout number.

Once a partition is specified, a start system with the number of solutions equal to the m-homo-geneous Bezout number can be computed. The general procedure for start system generation can be found in [30].

After generating the start system, the second step is to follow a finite number of solution paths to obtain the solutions of the target system. Suppose that the target system is F(x) and that the start system is G(x). Then the polynomial system H(x,t) is defined as:

$$H(x,t) = c(1-t)^k G(x) + t^k F(x) \qquad (2)$$

where $x = (x_1, \cdots, x_n)$ is a complex n-tuple, $t \in [0,1]$, c is a randomly chosen complex number and k is a positive number (usually 1 or 2). The system H(x,t) is known as linear and quadratic homotopy when k equals 1 and 2, respectively. From the definition above, H(x,t)=0 is a system of n equations in n unknowns and an additional parameter t. It is obvious that H(x,0)=cG(x) and H(x,1)=F(x). The basic premise of the continuation method is that a small change of t would result in only a small change of x as a solution of the system of equations H(x,t)=0. That is, if x(t) is a solution of H(x,t)=0, then for a small increment Δt>0, x(t+Δt) is "near" x(t). One method of computing x(t+Δt) from x(t) is to use a predictor-corrector method and the details can be found in [28] and [29]. Each solution of the start system is a solution of H(x,t)=0 at t=0 and represents a solution path of H(x,t)=0 from t=0 to t=1. Each solution path can be traced independently by successive small increments of t until t equals 1. Note that while some of the paths will converge to finite solutions, some others will diverge to solutions at infinity. The solutions of our target system are those that converge finitely in the continuation method as t approaches 1.

The software used in this research is PHC, a publicly available general-purpose polynomial systems solver using continuation method [28], [29]. Detailed description of polynomial homotopy continuation methods can be found in [19], [28]-[30].

3 Problem Formulation

In this work, the relative position of links and joints in mechanisms and manipulators is described using the variant of DH notation introduced by Pieper and Roth [31]. In this formulation, the parameters a_i, α_i, d_i and θ_i are defined so that a_i is the length of link i, α_i is the twist angle between the axes of joints i and i+1, d_i is the offset along joint i and θ_i is the rotation angle about joint axis i as shown in Figure 1. When joint i is revolute, then a_i, α_i and d_i are constants and are called structural parameters, while the value for θ_i depends on the configurations and is called a joint variable.

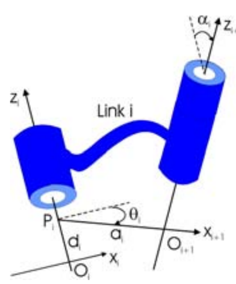

Figure 1 Denavit and Hartenberg Parameters

Reference frame R_i is attached at link i and its origin O_i is the intersection point of the common perpendicular between axes i and i-1 with joint axis i. Unit vector $\boldsymbol{z_i}$ of frame R_i is along joint axis i, unit vector $\boldsymbol{x_i}$ is along the common perpendicular of joint axes i and i-1. Positive directions for $\boldsymbol{x_i}$ and $\boldsymbol{z_i}$ are arbitrarily selected. (Note: letters in bold indicate vectors and matrices.) The homogeneous transformation matrix $\boldsymbol{A_i}$ that relates reference frame R_{i+1} to R_i, and its inverse matrix $\boldsymbol{A_i^{-1}}$ are given by:

$$\boldsymbol{A_i}=\begin{pmatrix} c_i & -s_i c_{\alpha_i} & s_i s_{\alpha_i} & a_i c_i \\ s_i & c_i c_{\alpha_i} & -c_i s_{\alpha_i} & a_i s_i \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{A_i^{-1}}=\begin{pmatrix} c_i & s_i & 0 & -a_i \\ -s_i c_{\alpha_i} & c_i c_{\alpha_i} & s_{\alpha_i} & -d_i s_{\alpha_i} \\ s_i s_{\alpha_i} & -c_i s_{\alpha_i} & c_{\alpha_i} & -d_i c_{\alpha_i} \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (3)$$

where: $c_i=\cos(\theta_i)$, $s_i=\sin(\theta_i)$, $c_{\alpha i}=\cos(\alpha_i)$ and $s_{\alpha i}=\sin(\alpha_i)$.

Consider the three-link open loop spatial chain with revolute (R) joints shown in Figure 2. Two frames are selected arbitrarily: a fixed reference frame R_0 and a moving end-effector frame R_e. Frame R_e will be defined in four distinct spatial locations. In addition to the three links of the manipulator, a stationary virtual link 0 is also assumed between axis $\boldsymbol{z_0}$ of frame R_0 and the first revolute joint axis. Frames are defined at each link using the DH procedure described above. Frame R_1 that is stationary is defined attached at link 0 having its $\boldsymbol{z_1}$ axis along the first revolute joint and its $\boldsymbol{x_1}$ axis along the common perpendicular of $\boldsymbol{z_0}$ and $\boldsymbol{z_1}$. Frame R_{i+1} is attached at the tip of link i (where i=1, 2, 3). The axis $\boldsymbol{z_4}$ is coincident with the axis $\boldsymbol{z_e}$ of the end-effector frame. The axis $\boldsymbol{x_4}$ is defined along the common perpendicular of $\boldsymbol{z_3}$ and $\boldsymbol{z_e}$ and the origin O_4 of R_4 is the point of intersection of $\boldsymbol{z_e}$ with its common perpendicular with $\boldsymbol{z_3}$. So frames R_4 and R_e have the same z-axis.

The homogeneous transformation matrices $\boldsymbol{A_i}$, with i=0, 1, 2, 3 describe frame R_{i+1} relative to R_i. The homogeneous transformation matrix $\boldsymbol{A_c}$ relates R_e to R_4. The relationship between these frames is a screw displacement: a rotation ϕ around the $\boldsymbol{z_4}$ axis and a translation d along the $\boldsymbol{z_4}$ axis. Homogeneous transformation matrix $\boldsymbol{A_h}$ relates directly the end-effector reference frame R_e to the frame R_0. Matrices $\boldsymbol{A_c}$ and $\boldsymbol{A_h}$ are written as:

$$\boldsymbol{A_c}=\begin{pmatrix} c_\phi & -s_\phi & 0 & 0 \\ s_\phi & c_\phi & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \boldsymbol{A_h}=\begin{pmatrix} l_1 & m_1 & n_1 & x_d \\ l_2 & m_2 & n_2 & y_d \\ l_3 & m_3 & n_3 & z_d \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (4)$$

where $\boldsymbol{l}=[l_1, l_2, l_3]^T$, $\boldsymbol{m}=[m_1, m_2, m_3]^T$, and $\boldsymbol{n}=[n_1, n_2, n_3]^T$, are the 3 by 1 vectors of the direction cosines of R_e in R_0. The parameters x_d, y_d, and z_d are the coordinates of the origin of R_e in R_0.

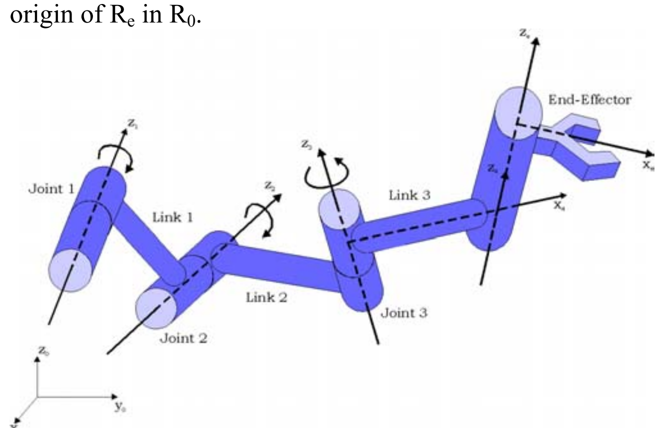

Figure 2 Schematic of a 3R Open loop Spatial Manipulator

An important feature in the matrix definition above is the use of matrix $\boldsymbol{A_c}$. In general, six parameters are needed to describe one reference frame relative to another. The DH parameterization uses only four parameters to describe the relative location between any two successive joint axes. Therefore, at the end-effector of the serial kinematic chain, the two additional parameters in $\boldsymbol{A_c}$ are used to completely specify the spatial position and orientation of the end-effector.

The loop closure equation of the manipulator is used to obtain the design equations:

$$\boldsymbol{A_0 A_1 A_2 A_3 A_c = A_h} \tag{5}$$

Equation (5) is a 4 by 4 matrix equation that results in six independent scalar equations. The right side of Equation (5), i.e. the elements of matrix $\boldsymbol{A_h}$, is known since they represent the position and orientation of frame R_e at each precision point. The left side of Equation (5) contains all the unknown geometric parameters of the manipulator which are the DH parameters a_i, α_i, d_i and θ_i for i=0, 1, 2, 3, and parameters ϕ and d of matrix $\boldsymbol{A_c}$. Joint angles θ_1, θ_2 and θ_3 have a different value for each precision point while all other 15 geometric parameters are constant. Thus for five precision points there are 30 unknown parameters in total, and there are 30 scalar equation that are obtained. Therefore, the maximum number of precision points for exact synthesis is five. For four precision points synthesis, which is studied in this paper, there are 27 unknowns (15 structural parameters and 12 joint variables) and 24 scalar equations, thus we can select three structural parameters arbitrarily as free choices.

Due to the arbitrary selection of the positive direction of $\boldsymbol{z_i}$ there will be two values for the twist angle, i.e. α_i and $\alpha_i+\pi$, that correspond to the same joint axes i and i+1. Similarly, due to the arbitrary selection of the positive direction of $\boldsymbol{x_i}$, there will be two values for the joint angle, i.e. θ_i and $\theta_i+\pi$, that describes the angle between $\boldsymbol{x_i}$ and $\boldsymbol{x_{i+1}}$. The consequence is that in this problem, where angles α_i and θ_i are calculated, both values for each one of these parameters will appear among the set of solutions. Obviously, only one of these values will be retained because they correspond to the same set of axes.

4 Design Equations at Each Precision Point

Using the loop closure equation of the manipulator (Equation 5), six scalar design equations are obtained at each precision point. The unknowns in these equations are the manipulator constant structural parameters and the joint variables θ_1, θ_2 and θ_3, which vary from precision point to precision point. To simplify the solution process, we eliminate the joint variables from the design equations at each precision point. Once the joint variables are eliminated, the new set of equations contains only unknowns that do not change from precision point to precision point. In this way, for each new precision point that is defined, new equations are added that have exactly the same structure as for the first precision point. In this section we present the method to obtain design equations devoid of the joint variables.

From Equation (3), it can be seen that the 3rd and 4th columns of matrix $\boldsymbol{A_i^{-1}}$ are independent of joint angle θ_i. Therefore, if Equation (5) is written as:

$$\boldsymbol{A_1 A_2 = A_0^{-1} A_h A_c^{-1} A_3^{-1}} \tag{6}$$

then the scalar equations that are obtained by equating the left and right side of the third and fourth columns of matrix Equation (6) will be devoid of joint angle θ_3.

From the third column of Equation (6), three scalar equations are obtained:

$$s_{\alpha_2} c_1 s_2 - c_{\alpha_1} s_{\alpha_2} s_1 c_2 + s_{\alpha_1} c_{\alpha_2} s_1 = c_0 L_1 + s_0 L_2 \tag{7}$$

$$s_{\alpha_2} s_1 s_2 - c_{\alpha_1} s_{\alpha_2} c_1 c_2 - s_{\alpha_1} c_{\alpha_2} c_1 = -c_{\alpha_0} s_0 L_1 + c_{\alpha_0} c_0 L_2 + s_{\alpha_0} L_3 \tag{8}$$

$$-s_{\alpha_1} s_{\alpha_2} c_2 + c_{\alpha_1} c_{\alpha_2} = s_{\alpha_0} s_0 L_1 - s_{\alpha_0} c_0 L_2 + c_{\alpha_0} L_3 \tag{9}$$

where $L_i = l_i A + m_i B + n_i C$, with i = 1, 2, 3, and $A=s\phi s\alpha_3$, $B= c\phi s\alpha_3$ and $C=c\alpha_3$ with $c\phi=\cos(\phi)$ and $s\phi=\sin(\phi)$.

From the fourth column of Equation (6), another three scalar equations are obtained:

$$\begin{gathered} a_2 c_1 c_2 - a_2 c_{\alpha_1} s_1 s_2 + a_1 c_1 + d_1 s_{\alpha_1} s_1 = \\ c_0 (M_1 + x) + s_0 (M_2 + y) - a_0 \end{gathered} \tag{10}$$

$$\begin{gathered} a_2 s_1 c_2 + a_2 c_{\alpha_1} c_1 s_2 - d_2 s_{\alpha_1} c_1 + a_1 s_1 = \\ -c_{\alpha_0} s_0 (M_1 + x) + c_{\alpha_0} c_0 (M_2 + y) + s_{\alpha_0} (M_3 + z) - d_0 s_{\alpha_0} \end{gathered} \tag{11}$$

$$\begin{gathered} a_2 s_{\alpha_1} s_2 + d_1 + d_2 c_{\alpha_1} \\ = s_{\alpha_0} s_0 (M_1 + x) - s_{\alpha_0} c_0 (M_2 + y) + c_{\alpha_0} (M_3 + z) - d_0 c_{\alpha_0} \end{gathered} \tag{12}$$

where: $M_i = l_i P + m_i Q + n_i R$, with i = 1, 2, 3 and $P=-a_3 c\phi - d_3 s\phi s\alpha_3$, $Q= a_3 s\phi - d_3 c\phi s\alpha_3$ and $R=-d_3 c\alpha_3 - d$. Note that Equations (9) and (12) are free of θ_1, thus, c_2 and s_2 can be computed by these two equations and their analytical expressions are free of θ_1 also. Using this result, θ_2 is essentially eliminated, for c_2 and s_2 can be eliminated from any equation by substituting the above result.

The final step is to obtain equations free of θ_1. To obtain such equations, we will consider the matrix Equation (6) again, written here as,

$$\boldsymbol{A_L = A_R} \tag{13}$$

where $\boldsymbol{A_L = A_1 A_2}$ and $\boldsymbol{A_R = A_0^{-1} A_h A_c^{-1} A_3^{-1}}$.

We will denote the third column vector of $\boldsymbol{A_L}$ and $\boldsymbol{A_R}$ as $\boldsymbol{U_L}$ and $\boldsymbol{U_R}$, respectively, and the fourth column vector of $\boldsymbol{A_L}$ and $\boldsymbol{A_R}$ as $\boldsymbol{V_L}$ and $\boldsymbol{V_R}$, respectively (Note: vectors $\boldsymbol{U_L}$, $\boldsymbol{U_R}$, $\boldsymbol{V_L}$ and $\boldsymbol{V_R}$ are 3 by 1; i.e. we neglect the fourth component which is the homogeneous coordinate). Then, we form the following three vector equations:

$$\boldsymbol{U_L \cdot V_L = U_R \cdot V_R} \tag{14}$$

$$\boldsymbol{V_L \cdot V_L = V_R \cdot V_R} \tag{15}$$

$$\boldsymbol{U_L \times V_L = U_R \times V_R} \tag{16}$$

Equations (14), (15) and (16) give a total of five scalar equations. For Equation (16), only the third component is used, i.e.

$$\mathbf{U_L(1) V_L(2) - U_L(2) V_L(1) = U_R(1) V_R(2) - U_R(2) V_R(1)} \tag{17}$$

It was found that Equations (14), (15) and (17) are naturally devoid of θ_1. With θ_2 eliminated by using the expressions of c_2 and s_2 calculated from Equations (9) and (12), the three equations are free of θ_1, θ_2 and θ_3 and have, respectively, the following form:

$$\sum_{X_j, X_k \in W} f_{X_j, X_k}(\alpha_0, \theta_0, \alpha_1) X_j X_k = 0 \tag{18}$$

$$\sum_{X_j, X_k \in W} g_{X_j, X_k}(\alpha_0, \theta_0, \alpha_1) X_j X_k = 0 \tag{19}$$

$$\sum_{X_j, X_k \in W} h_{X_j, X_k}(\alpha_0, \theta_0, \alpha_1) X_j X_k = 0 \quad (20)$$

Where W={F, G, H, S, P, Q, R, d_2, a_0, a_1, d_0, d_1, 1} and F=λA, G=λB, H=λC, S=$\lambda c\alpha_2$ and $\lambda=a_2/s\alpha_2$.

Note that Equations (18), (19) and (20) depend also on the parameters l_i, m_i, n_i, (i=1, 2, 3) and x, y and z, which are defined at each precision point and vary from precision point to precision point. Therefore, each precision point contributes three design equations (i.e. Equation (18), (19) and (20)), which are devoid of the joint variables and have as unknowns only the 15 constant structural parameters.

5 Solution Procedure Using Polynomial Continuation

In this paper we solve the geometric design problem of 3R manipulators with four precision points. In this case, there are 12 scalar equations in 15 unknown. This means that we can select three design parameters as free choices so that a well-determined system of twelve equations in twelve unknowns is obtained. In this paper, two different ways for selecting free choices have been considered. Both type of free choice selection involves assigning parameters to the base of the manipulator. The design equations for both types of selections are obtained by substituting the free choices made into Equations (18), (19) and (20).

5.1 Type 1 of Free Choice Selection

In this type, the free choices made are parameters α_0, a_0 and θ_0. By arbitrarily selecting the values for these parameters the designer selects the direction and partially locates the first joint of the manipulator with respect to a fixed reference frame.

After substituting the values of the free choices into Equations (18), (19) and (20), they become:

$$\sum_{X_j, X_k \in T_1} f_{X_j, X_k}(\alpha_1) X_j X_k = 0 \quad (21)$$

$$\sum_{X_j, X_k \in T_1} g_{X_j, X_k}(\alpha_1) X_j X_k = 0 \quad (22)$$

$$\sum_{X_j, X_k \in T_1} h_{X_j, X_k}(\alpha_1) X_j X_k = 0 \quad (23)$$

where: T_1={F, G, H, S, P, Q, R, d_0, d_1, a_1, d_2, 1}.

Note that f, g, h are polynomial functions of $c\alpha_1$ and $s\alpha_1$, which are transcendental functions in α_1. We can regard $c\alpha_1$ and $s\alpha_1$ as independent variables with an additional constraint equation $c\alpha_1^2 + s\alpha_1^2 - 1 = 0$. By incorporating the constraint equations, the new system is a multivariate polynomial system with thirteen equations in thirteen unknowns from $T_1 \cup \{c\alpha_1, s\alpha_1\}$.

Using a 2-partition G_1={F, G, H, S, P, Q, R, d_0, d_1, a_1, d_2} and G_2={$c\alpha_1$, $s\alpha_1$}, the multi-homogeneous bound is found to be 53248 after linear reduction by subtracting Equations (21), (22), (23) at the first, second and third precision points by the corresponding equations at the forth precision point.

Using PHC, a continuation method based on this 2-homogeneous number is employed and the numerical values of the variables in $G_1 \cup G_2$ are computed. The DH parameters of the design solutions are computed using a back-substitution procedure. It is found that out of 53248 paths, only 144 paths converge to true solutions of the design problem, the remaining paths are solutions at infinity and extraneous solutions. These 144 solutions are all numerically different but contain only 36 geometrically distinct solutions, where each geometrically distinct solution has four equivalent different representations in terms of DH parameters (see last paragraph of Section 3).

Therefore, at the most there are thirty-six distinct manipulators that can place their end-effectors in the four precision points specified by the designer in this case.

5.2 Type 2 of Free Choice Selection

In this type, the free choices made are d_0, α_0 and θ_0. The design equations (18), (19) and (20) become:

$$\sum_{X_j, X_k \in T_2} f_{X_j, X_k}(\alpha_1) X_j X_k = 0 \quad (24)$$

$$\sum_{X_j, X_k \in T_2} g_{X_j, X_k}(\alpha_1) X_j X_k = 0 \quad (25)$$

$$\sum_{X_j, X_k \in T_2} h_{X_j, X_k}(\alpha_1) X_j X_k = 0 \quad (26)$$

where T_2={F, G, H, S, P, Q, R, a_0, d_1, a_1, d_2, 1}.

Together with the constraint equation $c\alpha_1^2 + s\alpha_1^2 - 1 = 0$, the new system is again a multivariate polynomial system of thirteen equations in thirteen unknowns $T_2 \cup \{c\alpha_1, s\alpha_1\}$.

Using a 2-partition G_1={F, G, H, S, P, Q, R, a_0, d_1, a_1, d_2} and G_2={$c\alpha_1$, $s\alpha_1$}, after linear reduction, the 2-homogeneous number is again 53248.

As in type 1, a continuation method using PHC based on this 2-homogeneous number gives 144 numerically distinct solutions and only 36 geometrically distinct solutions.

Due to space limitations, numerical examples could not be included in this paper.

6 Conclusions

In this paper, the geometric design problem of serial-link spatial robot manipulators with three revolute (R) joints when four precision points are specified is solved using a polynomial homotopy continuation method. Four spatial positions and orientations are defined and the DH parameters of the 3-R manipulator are computed so that the manipulator will be able to place its end-effector at these four pre-specified locations. Two types of free choices selections are considered. It is shown that for both types of free choice selection, thirty-six manipulators can be found at the most that can place their end-effectors at the four specified precision points.

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