USE OF MATHEMATICAL PROGRAMMING IN THE ANALYSIS OF CONSTRAINED AND UNCONSTRAINED INDUSTRIAL EXPERIMENTS

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Introduction  
In the analysis of industrial experimental results, trade-offs between the optimal settings for the response mean, the response variance, and their associated costs often are inevitable in practice. In such situations, any of several methods typically are used to set the mean response equal to its target value with the response variance minimized to the extent possible. Typically, however, trade-offs between the costs and nonconforming rate associated with the resultant factor settings are not rigorously or explicitly explored. In order to better understand and address these trade-offs, mathematical optimization methods can complement other approaches by helping to explore alternate solutions more directly and determine the optimal factor settings that minimize poor quality or total costs.

For example, in practice, a large percentage of industrial experiments are of the "nominal-the-best" type for which the studied response characteristic $Y$ is continuous with a target value $T_y$, upper specification limit $USL_y$, and lower specification limit $LSL_y$. Three general objectives in this type of problem are to identify the best process design that
either minimizes the nonconformance rate, minimizes the factor setting costs, or minimizes the total costs of nonconformance and production. This article illustrates approaches in such cases for integrating mathematical programming with experimental design in order to help identify the overall optimal settings, possibly subject to upper bounds on the nonconforming rate or total cost, and to help conduct sensitivity analysis in cases where accurate cost estimates are not easily obtainable. These approaches are compared to a more conventional method using an example adapted from practice.

A Conventional Approach: Minimize Variability and Adjust Mean to Target

In experimental situations for which both the response mean \( \mu_y \) and standard deviation \( \sigma_y \) are of concern, a conventional approach for selecting factor settings often is based on the following interrelated steps (e.g., see Refs. 1–3):

1. Design and conduct screening and response surface experiment(s).
2. Analyze results, identify significant mean and variance effects, and develop appropriate prediction equations for \( \hat{\mu}_y \) and \( \hat{\sigma}_y \).
3. Minimize the predicted response standard deviation \( \hat{\sigma}_y \) via appropriate settings for significant variance effects.
4. Set or adjust the predicted response mean \( \hat{\mu}_y \) to the target value \( T_y \), such as by using one or more “adjustment” factors or via optimization methods.
5. Set any insignificant factors at their lowest cost settings.

In the simplest case, assuming the first two activities have been conducted following either conventional screening and/or response surface experimental methods, factor settings then can be chosen heuristically in the remaining steps first by identifying those factors and interactions that significantly impact the response variance and setting these in such a way so as to achieve the absolute minimum possible estimated variance (within the experimental region). The response mean estimate often then is centered on target by appropriately setting some other “adjustment” factor (ideally one with a statistically significant result with respect to the mean but not with respect to variability). If such an adjustment factor (with no variance effect) is not available, then that with the largest estimated mean effect might be used for this purpose (alternatives are to use that with the smallest estimated variance effect or with the largest ratio of mean-to-variance estimated effects).

In either case, if after setting the first factor all the way to one or the other end points of its experimental range (i.e., the coded +1 or −1 settings) still more adjustment is needed in order to move the response average to target, then a second factor is used for further adjustment (e.g., that with either the second largest estimated mean effect, second smallest estimated variance effect, or second largest ratio), and so on until the predicted mean response is adjusted such that \( \hat{\mu}_y = T_y \). By following this general approach, the predicted mean iteratively can be set to its target value with the predicted variability usually minimized as much as possible (or close to it). Although not the focus here, other approaches include treating the mean and variance as multiple responses, using optimization or search algorithms to minimize the variance subject to the equality constraint that \( \hat{\mu}_y = T_y \), or maximizing a desirability function; for example, see Refs. 4 and 5. Also, note that confirmation experiments are recommended at and about the selected settings in order to verify their optimality.

An Example: Laser Stock Removal

As an example, consider a laser stock removal process in which the response of interest \( Y \) is a depth of cut (measured in millimeters), with approximately 160,000 units produced per month and the following three factors available for adjustment:

- Beam speed \( (x_1) \)
- Frequency \( (x_2) \)
- Power level \( (x_3) \)

In order to determine the best settings for speed, frequency, and power that meet a customer depth-of-cut requirement of \( Y = 800 \pm 40 \mu m \) most consistently and at minimum cost, each parameter can be set anywhere between the following minimum and maximum values:

- Beam speed must be between 4 in./min (−1) and 16 in./min (+1).
- Frequency must be between 2000 Hz (−1) and 3000 Hz (+1)
- Power must be between 5 W (−1) and 10 W (+1)

with −1 and +1 indicating the coded settings of each process parameter. These experimental conditions are summarized in Table 1, along with estimated relative monthly costs of operating at each factor setting, based on historical experience and assumed here to be linear. Note that conducting sensitivity analysis is especially important if such costs
cannot be estimated very precisely. For illustration purposes, assume that a series of experiments resulted in the regression coefficients (or half-effects) for mean and standard deviation summarized in Table 2, with no evidence of nonlinearity, and the following prediction equations for the response mean \( \hat{\mu}_y \) and standard deviation \( \hat{\sigma}_y \):

\[
\hat{\mu}_y = 750 + 40x_1 + 25x_2 + 10x_3 - 2x_1x_2
\]

and

\[
\hat{\sigma}_y = 10.5 + x_1 - 2x_2 + 0.5x_2x_3.
\]

Using these results, the heuristic approach outlined earlier first produces \( x_1 = -1, x_2 = +1 \), and \( x_2x_3 = -1 \), and thus \( x_3 = -1 \) in order to minimize \( \hat{\sigma}_y \). Because these settings result in an estimated mean of

\[
\hat{\mu}_y = 750 + 40(-1) + 25(+1) + 10(-1) - 2(-1)(+1) = 725 < T_y = 800,
\]

with \( x_1 = -1 \) making the largest negative contribution to the mean, then setting

\[
T_y = 800 = \hat{\mu}_y = 750 + 40x_1 + 25(1) + 10(-1) - 2x_1(+1) = 765 + x_1(40 - 2)
\]

and solving for \( x_1 \) yields

\[
x_1 = \frac{800 - 765}{38} = \frac{35}{38} \approx 0.9211.
\]

Thus, the final recommended factor settings using this approach are

- Speed \( (x_1) = \frac{35}{38} \approx 0.92 \) (15.48 in./min)
- Frequency \( (x_2) = 1.00 \) (3000 Hz)
- Power \( (x_3) = -1.00 \) (5 W),

which in this case also agree with results obtained via optimization as shown later (i.e., minimizing variance subject to \( \hat{\mu}_y = T_y \)).

Some Comments

As stated earlier, note that this approach results in factor settings that yield mean and standard deviation estimates of

\[
\hat{\mu}_y = 750 + 40\left(\frac{35}{38}\right) + 25(1) + 10(-1) - 2\left(\frac{35}{38}\right)(1) = 800 \text{ mm},
\]

which is the desired target value, and

\[
\hat{\sigma}_y = 10.5 + \left(\frac{35}{38}\right) - 2(1) + 0.5(1)(-1) = 8.92 \text{ mm},
\]

which is only moderately greater than the minimum possible predicted standard deviation of

\[
\min \hat{\sigma}_y = 10.5 + (-1) - 2(1) + 0.5(1)(-1) = 7 \text{ mm}.
\]

Thus, the minimum possible estimated variance has been achieved subject to the requirement that the estimated process mean be centered equal to its target value. Note that in this example, these same factor settings and prediction equations would be produced by the largest ratio of effects (LRE) approach, whereas the smallest variance effect (SVE) approach here produces the different results of \( x_1 = \frac{35}{38} \approx 0.9347, x_2 = 1, x_3 = 1 \), \( \hat{\mu}_y = 800 \text{ mm} \), and \( \hat{\sigma}_y = 9.35 \text{ mm} \), which is larger than the minimum possible standard deviation of 8.92 mm found earlier given \( \hat{\mu}_y = 800 \text{ mm} \). Moreover, in any case, the nonconforming rate and costs associated with these results are typically only considered in passing and without much formal quantitative exploration of alternatives. It also is important to comment that none of these approaches necessarily result in the minimum expected
number of nonconforming parts per million (NCPPM) or the
overall minimum cost, unless all factors used for adjustment-
to-target do not at all affect the response variance and all
significant factors have negligible cost differences across
their experimental ranges; both of these conditions often not
being the case in practice implicitly results in a quality–cost
compromise.

For example, in the above case and assuming normality
of the response \( Y \), the estimated mean NCPPM correponds-
ing to the identified factor settings under the largest-mean-
effect (LME) and smallest-variance-effect (SVE) approaches
are approximately 7.34 and 20.47, respectively (with process
capability index estimates of approximately \( \hat{C}_{pk} = 1.49 \) and
1.42, respectively), which we will see can be slightly im-
proved. Also, note that using the costs shown in Table 1 and
assuming linear costs and linear factor effects, the total factor
operating cost corresponding to these factor settings can be
seen to be

\[
\text{FC} = \text{Total production factor costs} = \sum_{\text{Factor } i} \left( \frac{(\text{Cost at factor } i \text{ midpoint}) + (\text{Cost half-range for Factor } i)}{2} \times (\text{Coded setting for factor } i) \right)
\]

\[
= \left( \frac{\$400 + \$200}{2} + \frac{\$400 - \$200}{2} \right) x_i + \left( \frac{\$800 + \$200}{2} + \frac{\$800 - \$200}{2} \right) x_j + \left( \frac{\$200 + \$100}{2} + \frac{\$200 - \$100}{2} \right) x_k
\]

\[
= (\$300 + \$100 x_i) + (\$500 + \$300 x_j) + (\$150 + \$50 x_k)
\]

\[
= \$950 + \$100 x_i + \$300 x_j + \$50 x_k
\]

\[
= \$950 + \$100 \left( \frac{35}{38} \right) + \$300(1) + \$50(-1)
\]

\[
= \$1292.11 \text{ per month},
\]

where \( C_i^+ \) and \( C_i^- \) represent the relative costs of each factor
\( i \) at its high and low setting, respectively. As an aside, note
that a little algebra simplifies this cost expression to

\[
\text{FC} = \frac{1}{2} \left[ \sum_{i} C_i^+ + \sum_{i} C_i^- + \sum_{i} (C_i^+ - C_i^-) x_i \right]
\]

\[
= \frac{1}{2} \left[ \text{Sum of maximum costs} + \text{Sum of minimum costs} + \sum_{\text{Factor } i} (\text{Full cost range}) (\text{Setting}_i) \right]
\]

\[
= \frac{1}{2} \left[ \$1400 + \$500 + \$200 \left( \frac{35}{38} \right) + \$600(1) + \$100(-1) \right]
\]

\[
= \$1292.11,
\]

as before, which is significantly greater than the minimum
possible total factor cost of

\[
\text{FC} = \sum_{i} C_i^- = \text{Sum of minimum costs}
\]

\[
= \$200 + \$200 + \$100 = \$500 \text{ per month}.
\]

Therefore, this may not be necessarily the best approach in
order to achieve the best compromise between the dual ob-
jectives of the highest possible quality at the lowest possible
cost (in this case achieving neither). As an alternate ap-
proach to address these considerations more explicitly, after
response mean and variance equations have been developed,
mathematical programming models then can be used to help
identify optimal factor settings and to explore trade-offs
between operating costs and nonconformance rates. Several
such possible uses are illustrated in the following sections,
dependent on the primary experimental objective and any
external cost or quality constraints. (Also, see Refs. 6–8 for
examples of other uses of mathematical optimization in ex-
perimental design scenarios.)

**Model 1: Minimize Nonconformance Rate**

If the primary objective is to minimize the nonconform-
ing rate, possibly subject to some specified upper bound on
acceptable production costs (i.e., the total operating costs
of all factor settings), then the following general nonlinear
programming (NLP) formulation can be used:

\[
\text{Minimize: Estimated nonconformance rate}
\]

\[
\text{subject to:}
\]

1. Response mean and standard deviation prediction
equations
2. Response upper and lower specification limits
3. Possible upper bound on total factor setting costs
4. Factor costs at upper and lower settings
5. All factor settings between their coded \(-1\) and \(+1\)
settings

This model will identify the combination of factor settings
that together minimize the predicted nonconforming rate
subject to the above five sets of constraints. For example,
using the following notation and assuming the response \( Y \) is
normally distributed,

\[
C_i^+ = \text{the relative cost of factor } i \text{ at its high setting}
\]

\[
C_i^- = \text{the relative cost of factor } i \text{ at its low setting}
\]

\[
\text{USL}_Y = \text{upper specification limit for the response } Y
\]
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\[ \text{LSL}_y = \text{lower specification limit for the response } Y \]
\[ \Phi(z) = P(Z < z), \text{where } Z \sim N(0, 1) \]
\[ \text{NR}_y = \text{estimated nonconforming rate for the response } Y \]
\[ = P\left(Z < \frac{\text{LSL}_y - \mu_y}{\sigma_y}\right) + P\left(Z > \frac{\text{USL}_y - \mu_y}{\sigma_y}\right) \]
\[ = 1 - \Phi\left(\frac{\text{USL}_y - \mu_y}{\sigma_y}\right) + \Phi\left(\frac{\text{LSL}_y - \mu_y}{\sigma_y}\right) \]
\[ \text{FC} = \text{the total factor cost of all factors at their given settings} \]

then in the laser removal example, if the total operating cost cannot be greater than, for example, $1100, the above model becomes

\[ \text{Minimize: } 1 - \Phi\left(\frac{\text{USL}_y - \mu_y}{\sigma_y}\right) + \Phi\left(\frac{\text{LSL}_y - \mu_y}{\sigma_y}\right) \]
\[ \text{subject to:} \]
\[ \mu_y = 750 + 40x_1 + 25x_2 + 10x_3 - 2x_1 x_2 \]
\[ \sigma_y = 10.5 + x_1 - 2x_1 + 0.5x_1 x_3 \]
\[ \text{FC} = \sum_{i=1}^{\infty} \left(\frac{C_i^+ + C_i^-}{2}\right) x_i \]
\[ \text{FC} \leq 1100 \]
\[ \text{LSL}_y = 760, \quad \text{USL}_y = 840 \]
\[ C_1^- = 200, \quad C_1^+ = 400 \]
\[ C_2^- = 200, \quad C_2^+ = 800 \]
\[ C_3^- = 100, \quad C_3^+ = 200 \]
\[ x_i \geq -1 \text{ for } i = 1, 2, 3 \]
\[ x_i \leq +1 \text{ for } i = 1, 2, 3. \]

Note that although several of the above constraints could be combined to make the formulation slightly more efficient computationally (e.g., all inequality and equality cost constraints could be combined into the simpler requirement that $950 + 100x_1 + 300x_2 + 500x_3 \leq 1100$), it can be left in the above form with negligible loss in efficiency for easier readability, comprehension, revision, and sensitivity analysis. Also note that prediction equations for the mean \( \mu_y \) and standard deviation \( \sigma_y \) are included as equality constraints (rather than integrated into the objective function) as a matter of preference for the same reason, although this must not necessarily be the case.

The above optimization problem can be solved using any standard NLP software (e.g., LINDO, GAMS, CPLEX, and others), as well as optimization modules included in many spreadsheets (such as Microsoft Excel’s, Lotus 1-2-3’s, or Quattro Pro’s Solvers). These programs use any of several optimization algorithms (9–11) to identify the optimal parameter settings that will result in the minimum value for the objective function (here, the nonconformance rate) while still satisfying all stated constraints. In the present example, solution of the above model results in a minimum objective function value (nonconformance rate) of 0.000498 (or an estimated 498 mean NCPPM) at the following optimal settings for the decision variables (factors):

- Speed \((x_1^*) = 0.9834 (15.90 \text{ in./min})\)
- Frequency \((x_2^*) = 0.0055 (2502.77 \text{ Hz})\)
- Power \((x_3^*) = 1.00 (10 \text{ W})\)

These optimal factor settings result in an estimated response mean and standard deviation of

\[ \mu_y^* = 750 + 40(0.9834) + 25(0.0055) \]
\[ + 10(1) - 2(0.9834)(0.0055) = 799.46 \text{ mm} \]

and

\[ \sigma_y^* = 10.5 + 0.9834 - 2(0.0055) + 0.5(0.0055)(1) \]
\[ = 11.475 \text{ mm}, \]

which equates to a process capability estimate of \( C_{p,*} \approx 1.15 \) and a total operating factor cost equal to its upper bound \( \text{FC}^* = 950 + 100x_1 + 300x_2 + 500x_3 \)
\[ = 950 + 100(0.9834) + 300(0.0055) + 500(1) \]
\[ = 1910 \text{ per month.} \]

Note that, in practice, the total factor cost FC almost always will increase to its permitted upper bound in order to reduce NCPPM as close as possible to its absolute minimum (unless the total factor cost of all settings that together yield the absolute minimum possible NCPPM is less than this constraint value). Alternatively, if an upper limit on the total factor cost does not exist or is not well defined, then this constraint can be set arbitrarily high, removed altogether, or iteratively adjusted via sensitivity analysis in order to explore a range of cost–quality trade-offs, as illustrated later in the Discussion section. In the present scenario, for example, setting the right-hand side of the factor cost constraint equal to or greater than the sum of the maximum individual factor costs,

\[ \text{FC} \leq \sum_i C_i^+ = 400 + 800 + 200 = 1400, \]

essentially results in an unconstrained minimum possible estimated mean NCPPM of 7.29 at a total factor cost of $1291.49. Also note that in the current example, this optimization formulation happens to result in the predicted response mean being nearly equal to the target value halfway between the lower and upper specification limits (here, \( \mu_y^* = 799.46 \) versus \( T_y = 800 \)), although this must not always be the case if one or more factors that impact the re-
response mean have considerably higher costs relative to those impacting the response variance.

As an implementation aside, note that because the prediction equations and objective function are nonlinear, some caution should be taken to avoid local suboptimization (such as via the historical practice of verifying results from several different starting conditions), especially if using spreadsheet solvers. Additionally, most commercial spreadsheets have built-in normal cumulative density function statements that can be used in the above objective function, whereas if traditional optimization software is used, then a closed-form CDF approximation can be substituted in the objective function or added as an equality constraint.

**Model 2: Minimize Factor Costs**

As a converse approach to the same problem, if the primary objective instead is to minimize the total factor costs, now possibly while holding the nonconforming rate or mean NCPPM below some specified level (e.g., 0.1% defective or 1000 NCPPM), then the following general NLP formulation can be used:

Minimize: Total factor costs
subject to:

1. Response mean and standard deviation prediction equations
2. Response upper and lower specification limits
3. Possible upper bound on nonconformance rate
4. Factor costs at upper and lower settings
5. All coded factor settings between -1 and +1

Using the same notation and assumptions as in Model 1, for the laser removal example this problem then becomes

Minimize: \[ \frac{1}{2} \left( C_1^+ + C_1^- + C_2^+ - C_2^- \right) \]

subject to:

- Mean: \[ \bar{\mu}_y = 750 + 40x_1 + 25x_2 + 10x_3 - 2x_1x_2 \]
- Standard deviation: \[ \bar{\sigma}_y = 10.5 + x_1 - 2x_2 + 0.5x_1x_2 \]
- Lower specification limit: \[ \text{LSL}_y = 760 \]
- Upper specification limit: \[ \text{USL}_y = 840 \]
- Nonconformance rate: \[ \bar{N}R_y = 1 - \Phi \left( \frac{\text{USL}_y - \bar{\mu}_y}{\bar{\sigma}_y} \right) + \Phi \left( \frac{\text{LSL}_y - \bar{\mu}_y}{\bar{\sigma}_y} \right) \]
- Nonconformance constraint: \[ \bar{N}R_y \leq 0.001 \]
- Cost constraints: \[ C_1^- = 200, \quad C_1^+ = 400 \]
- \[ C_2^- = 200, \quad C_2^+ = 800 \]
- \[ C_3^- = 100, \quad C_3^+ = 200 \]
- Factor settings: \[ x_i \geq -1 \quad \text{for} \quad i = 1, 2, 3 \]
- \[ x_i \leq +1 \quad \text{for} \quad i = 1, 2, 3 \]

Solution of this mathematical optimization problem results in the following optimal factor settings (decision variables):

- Speed \( (x_1^+) = 1.00 \) (16 in./min)
- Frequency \( (x_2^+) = -0.1473 \) (2426.37 Hz)
- Power \( (x_3^+) = 1.00 \) (10 W),

which result in the minimum possible total factor setting cost of

\[ \text{FC} = 950 + 100(1) + 300(-0.1473) + 50(1) = 1055.82 \text{ month.} \]

These settings, however, now result in an estimated response mean and standard deviation of

\[ \bar{\mu}_y^* = 750 + 40(1) + 25(-0.1473) + 10(1) - 2(1)(-0.1473) = 796.61 \text{ mm} \]

and

\[ \bar{\sigma}_y^* = 10.5 + (1) - 2(-0.1473) + 0.5(-0.1473)(1) = 11.72 \text{ mm}, \]

which increases the estimated nonconformance rate to 0.001 (or 1000 NCPPM) and decreases \( C_{nk}^* \) to approximately 1.04. Note that the implied resultant estimated nonconformance rate has been forced equal to its constraint upper bound, \( \bar{N}R_y = 0.001 \), in order to reduce the total factor costs as much as possible. (Generally, this will be the case unless the factor settings that minimize total operating costs result in a nonconformance rate less than that specified in the constraint.) Again by relaxing or tightening the FC or NCPPM constraint in either model, the trade-offs between these two performance criteria can be explored, as discussed and illustrated further in the Discussion section.

**Model 3: Minimize Expected Total Costs**

(Factor Setting Costs and Expected Nonconformance Costs)

As an alternative to either of the above models, if the cost per nonconforming item is known or can be estimated reasonably, then the total cost due to both the factor settings and the nonconformance rate can be combined into a more global optimization model with the single objective being to minimize total costs as follows:

Minimize: Estimated total costs
subject to:

1. Response mean and standard deviation prediction equations
2. Response upper and lower specification limits
3. Defect cost and factor cost equations
4. Factor costs at upper and lower settings
5. All coded factor settings between -1 and +1

By adding the predicted expected nonconformance cost to the factor setting costs in the objective function of Model 2, in our example this becomes

\[
\begin{align*}
\text{Minimize:} & \quad \sum_{i=1}^{5} \left( \frac{C_i^+ + C_i^-}{2} + \frac{C_i^+ - C_i^-}{2} x_i \right) \\
& \quad + \left[ 1 - \Phi\left( \frac{\text{USL}_y - \hat{\mu}_y}{\hat{\sigma}_y} \right) + \Phi\left( \frac{\text{LSL}_y - \hat{\mu}_y}{\hat{\sigma}_y} \right) \right] c_D V \end{align*}
\]

subject to:

\[
\begin{align*}
\hat{\mu}_y &= 750 + 40x_1 + 25x_2 + 10x_3 - 2x_1x_2 \\
\hat{\sigma}_y &= 10.5 + x_1 - 2x_2 + 0.5x_1x_3 \\
\text{LSL}_y &= 760, \quad \text{USL}_y = 840 \\
C_D &= \$1 \\
V &= 160,000 \\
C_i^+ &= 200, \quad C_i^- = 400 \\
C_i^+ &= 200, \quad C_i^- = 800 \\
C_i^+ &= 100, \quad C_i^- = 200 \\
x_i &\geq -1 \quad \text{for } i = 1, 2, 3 \\
x_i &\leq +1 \quad \text{for } i = 1, 2, 3
\end{align*}
\]

where \( C_D \) and \( V \) denote the relative cost per defective item (i.e., the average cost of not being within specifications) and the volume of items produced per time period, respectively. Again, note that this model can be thought of as a more global optimization model, as it captures both the costs of production (factor settings) and the associated costs of poor quality. Solution of this global model results in the following optimal settings for the decision variables:

- Speed (\( x_1^+ \)) = 1.00 (16 in./min)
- Frequency (\( x_2^+ \)) = -0.03496 (2482.52 Hz)
- Power (\( x_3^+ \)) = 1.00 (10 W)

and, thus, a total factor cost of

\[
\text{FC}^* = \$950 + \$100(1) + \$300(-0.03496) + \$50(1) = \$1089.51 \text{ per month.}
\]

These factor settings also produce an estimated response mean and standard deviation of

\[
\hat{\mu}_y^* = 750 + 40(1) + 25(-0.03496) + 10(1) - 2(1)(-0.03496) \approx 799.20 \text{ mm}
\]
and

\[
\hat{\sigma}_y^* = 10.5 + (1) - 2(-0.03496) + 0.5(-0.03496)(1) = 11.552 \text{ mm,}
\]

which result in an estimated nonconforming rate of 0.00055207 (or 552.07 NCPPM), an estimated process capability of \( \hat{C}_{pk} \approx 1.13 \), and a minimum predicted expected total cost of

\[
\hat{T}C = \text{Total factor cost} + \text{Predicted expected nonconformance cost} = \sum_{i=1}^{5} \left( \frac{C_i^+ + C_i^-}{2} + \frac{C_i^+ - C_i^-}{2} x_i \right) + \Phi\left( \frac{\text{USL}_y - \hat{\mu}_y}{\hat{\sigma}_y} \right) + \Phi\left( \frac{\text{LSL}_y - \hat{\mu}_y}{\hat{\sigma}_y} \right) c_D V
\]

\[
= (\$950 + \$100x_1 + \$300x_2 + \$50x_3)
\]

\[
+ \left[ 1 - \Phi\left( \frac{\text{USL}_y - \hat{\mu}_y}{\hat{\sigma}_y} \right) + \Phi\left( \frac{\text{LSL}_y - \hat{\mu}_y}{\hat{\sigma}_y} \right) \right] c_D V
\]

\[
= (\$1177.84 \text{ per month.})
\]

Of course, although not included in the earlier two optimization models, the associated total expected costs in each of these cases similarly can be computed for comparison purposes, as illustrated in the next section. The effect of the relative cost per defect on the expected cost also is explored in the following section.

### Discussion

Table 3 summarizes the results of each of the above approaches, also including the total estimated cost for each approach in order to facilitate comparison with the global model. As shown, each approach can produce important differences in factor settings, nonconforming rates, and costs. For example, the global minimum total cost model (Model 3) results in an overall total cost that is approximately 8.9% and 12.3% smaller than the conventional LME and SVE approaches, respectively. It also is important to note that the results of the first two (constrained) models are highly dependent on the value placed on the factor cost or nonconforming rate constraint (as can be seen by considering their unconstrained results shown in parentheses in Table 3).

### Sensitivity Analysis and Impact of Cost per Defect

The impact of the cost per defect relative to the factor costs can be explored by executing the total cost model (Model 3) for a variety of \( C_D \) values, which also can be
### Table 3. Summary of Results of Each Approach

<table>
<thead>
<tr>
<th>METHOD FOR SETTING FACTORS</th>
<th>OPTIMAL FACTOR SETTINGS</th>
<th>ESTIMATED RESULTS OF FACTOR SETTINGS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( x_1 ) (SPEED)</td>
<td>( x_2 ) (FREQ.)</td>
</tr>
<tr>
<td>Conventional SVE</td>
<td>0.3947</td>
<td>1</td>
</tr>
<tr>
<td>Conventional LME</td>
<td>0.9211</td>
<td>1</td>
</tr>
<tr>
<td>Conventional LRE</td>
<td>0.9211</td>
<td>1</td>
</tr>
<tr>
<td>Min. NCPPM (unconstrained)</td>
<td>0.9834</td>
<td>0.0055</td>
</tr>
<tr>
<td>Min. factor cost (unconstrained)</td>
<td>(0.9148)</td>
<td>(1)</td>
</tr>
<tr>
<td>Min. total costs</td>
<td>1</td>
<td>-0.1473</td>
</tr>
<tr>
<td>(unconstrained)</td>
<td>(-1)</td>
<td>(-1)</td>
</tr>
</tbody>
</table>

Note: SVE: Smallest variance effect; LME: largest mean effect; LRE: largest ratio of effects.
IMPORTANCE FOR SENSITIVITY ANALYSIS IF THIS COST IS NOT KNOWN WITH REASONABLE ACCURACY. AS SHOWN IN TABLE 4, FOR EXAMPLE, THE OPTIMAL POLICY IN THE LASER REMOVAL SCENARIO IS FAIRLY DEPENDENT ON THIS VALUE, WITH INCREASES IN \( C_D \) RESULTING IN SETTINGS THAT YIELD FEWER DEFECTS BUT AT THE (LESSER) EXPENSE OF HIGHER FACTOR COSTS, AS IS INTUITIVE. ALSO, NOTE THE PIVOTING OF FACTORS FROM THEIR HIGH OR LOW SETTINGS TO INTERMEDIATE VALUES. FOR EXAMPLE, INCREASING FROM \( 0 < C_D < 4 \) TO \( C_D \geq 16 \) IN THIS CASE EXCHANGES THE OPTIMAL SETTINGS FROM \( x_1 = 1 \) AND \( -1 < x_3 < 1 \) NOW TO \( -1 < x_1 < 1 \) AND \( x_3 = 1 \). IT IS ALSO INTERESTING THAT BEYOND THIS POINT, INCREASES ABOVE \( C_D \geq 16 \) IN THIS EXAMPLE ONLY IMPACT THE OPTIMAL SOLUTION MARGINALLY, AGAIN USEFUL KNOWLEDGE IF COSTS CAN ONLY BE ESTIMATED ROUGHLY TO BE WITHIN SOME GENERAL RANGE. SIMILAR SENSITIVITY ANALYSIS O}
Figure 1. Optimal pairwise frontier for minimum factor setting costs versus minimum estimated NCPPM (0–100,000).

Figure 2. Optimal pairwise frontier for minimum factor setting costs versus minimum estimated NCPPM (only for 0–5000 NCPPM).
MATHEMATICAL PROGRAMMING IN INDUSTRIAL EXPERIMENTS

Figure 3. Effect of estimated total cost (based on optimal NCPPM versus factor cost alone) (note log scale).

Figure 4. Optimal relationship between estimated NCPPM and total costs.
be seen in Figure 3, if the process is optimized based only on estimated factor costs $\hat{T}$ (i.e., Model 1 or 2), then increases in $C_D$ significantly increase the true estimated total cost $\hat{T}$. By optimizing on the total cost, however, the impact of increases or estimation errors in $C_D$ is reduced quite considerably, as shown in Figure 4. Thus, there clearly is significant value in using Model 3 even in cases for which the cost per defect cannot be estimated exactly, which is an interesting result.

More generally, integration of design of experiments and mathematical programming methods can complement traditional postexperiment approaches for setting factors, particularly when factor and nonconformance costs are of concern, as illustrated earlier, with spreadsheet implementation allowing sensitivity analysis to be conducted easily for different cost, factor settings, response distribution, and other assumptions. These approaches also can easily be adapted to allow for several extensions, including nonlinear factor costs, smaller-the-better or larger-the-better type problems, one-sided specifications or asymmetric targets, multiple responses, and the use of loss functions in place of specification-based nonconforming costs. Related multicriteria and goal programming optimization methods also could be used as alternatives to nonlinear programming models.

References


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