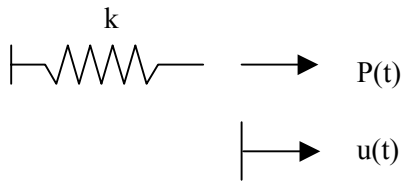
STRUCTURAL DYNAMICS

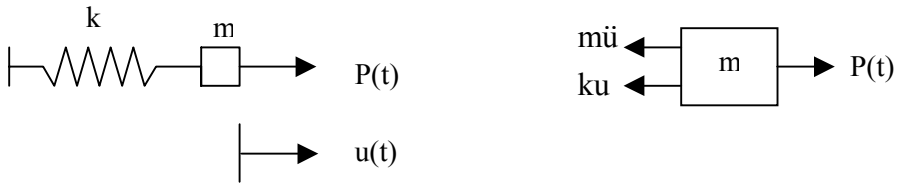
We can consider dynamic analysis as an extension of the tools of structural analysis to cases where inertia and damping forces can not be neglected. We note that the presence of forces that vary with time doesn't necessarily imply that one is dealing with a dynamic problem. For example, if we consider a massless spring subjected to a force P(t), as shown in the sketch,



the displacement is clearly given by;

$$u(t) = \frac{P(t)}{k}$$

As we can see this is not a "dynamic problem" but a succession of static problems. If we attach a mass at the end of the spring and subject the system to a load P(t) the situation one faces is the one depicted in the sketch.



From equilibrium considerations one can readily see that the force in the spring has contributions from the applied load and from the inertia in the mass. The spring elongation, therefore, is given by;

$$u(t) = \frac{P(t) - m\ddot{u}}{k} \qquad (1)$$

The previous equation would typically be written as;

$$m\,\ddot{u} + k\,u = P(t) \qquad (2)$$

Needless to say, the problem is a dynamic one since inertial forces are involved. In this class we examine techniques to formulate the equations of motion as well as the methods

used to solve them.

The Partial Differential Equation Formulation for Dynamic Problems

Since the movement of any material point is a function of time and the mass of a structure is distributed, the formulation of structural dynamic problems leads to partial differential equations. The essential features of the continuous formulation can be clarified by considering the case of a rod subjected to an arbitrary dynamic load at x = L.

Considering equilibrium of the differential element (in the horizontal direction) we get;

$$\frac{\partial^2 u(x,t)}{\partial t^2}(\rho A\, dx) + \sigma A - (\sigma + \frac{\partial \sigma}{\partial x} dx)A = 0 \qquad (3)$$

assuming the material behaves linearly we can express the stress as the product of the strain times the modulus of elasticity of the rod, E, namely;

$$\sigma = E\frac{\partial u(x,t)}{\partial x} \qquad (4)$$

substituting eq.4 into eq.3 and simplifying one gets;

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{E}{\rho}\frac{\partial^2 u(x,t)}{\partial x^2} = 0 \qquad (5)$$

which needs to be solved subject to the boundary conditions;

$$u(0,t) = 0 \qquad (6a)$$

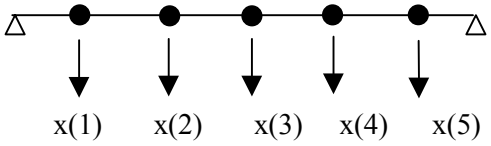
$$E\frac{\partial u(L,t)}{\partial x} = \frac{P(t)}{A} \qquad (6b)$$

The boundary condition in eq.6a is simply a statement that the rod is held at x = 0 while that in 6b indicates that the stress at the free end must equal the applied stress.

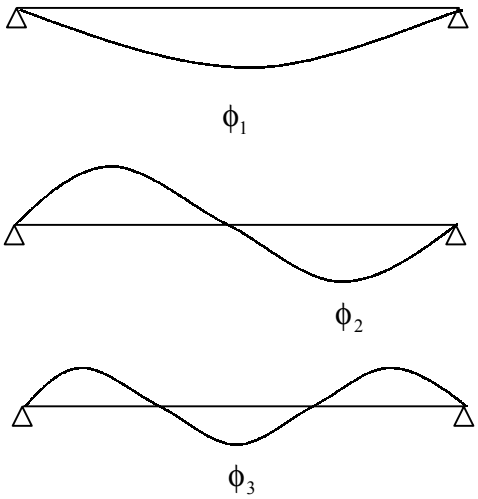
As the rod example shows, consideration of the distributed nature of the mass in a structure leads to equations of motion in terms of partial derivatives. Considerable simplification is realized, however, if the problem is formulated in terms of ordinary differential equations. The process of passing from the partial derivatives to ordinary differential equations is known as discretization. The concept of degree-of-freedom (DOF) is intimately associated with the process of discretization.

A degree of freedom is any coordinate used to specify the configuration of a physical system.

As noted, discretization converts the dynamic problem from one of partial differential equations to a set of ordinary differential equations. In these equations the unknowns are the temporal variation of the DOF. It is worth noting that the disctretization can be achieved by either physical lumping of the masses at disctrete locations.



or by using assumed modes shapes and treating the DOF as scaling parameters of the shapes. The idea is illustrated in the sketch, where we show 3 shapes that may be used to approximate the displacement field.



In particular, we would write;

$$u(t) = q_1(t)\phi_1 + q_2(t)\phi_2 + q_3(t)\phi_3 \qquad (7)$$

where the q_i's are the DOF. Calculating the time history of these q_i's is the object of the dynamic analysis for the discretized problem. In general, the assumed modes approach approximates the displacement field as;

$$u(x,y,z,t) = \sum_{n=1}^{N} q_n(t)\, \phi_n(x,y,z) \qquad (8)$$

where N = number of assumed shapes.

Finite Element Method

The FEM is a particular form of discretization where the assumed shapes do not cover the complete structure but are restricted to small portions of it (the FE).

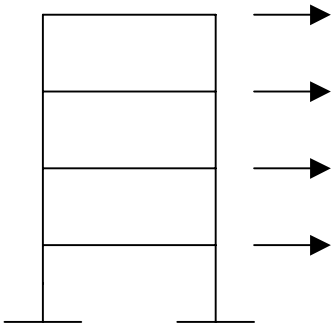
Temporal Discretization

If time is treated as a discrete variable then the ordinary differential equations that are obtained as a result of the spatial discretization become difference equations which can be solved with algebraic techniques. Discussion of various numerical techniques where time is treated as a discrete variable are presented in latter sections.

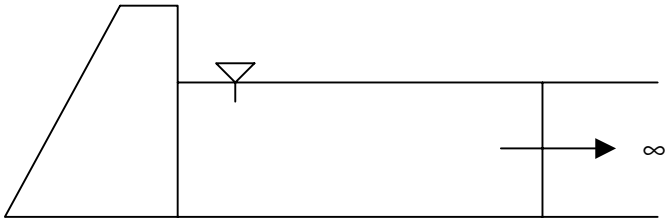
Types of Models:

It is worth noting that the problems we face may include:

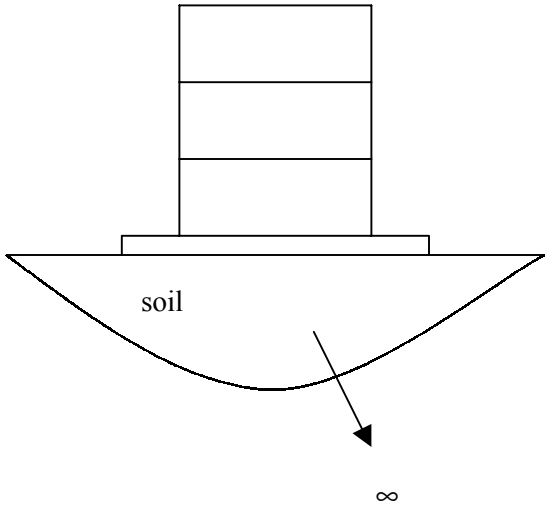
a) Models where the domain is finite.



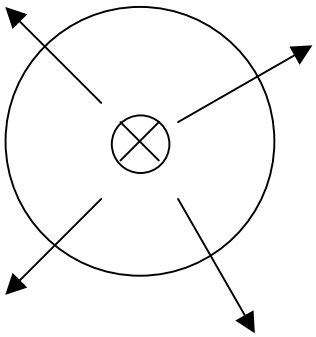
b) Models where the domain is semi-infinite.



or



c) Models where the domain is infinite (an example is a mine with an explosion or the computation of vibrations of the earth from a very deep earthquake)



System Classification

In the study of vibrations it is important to determine whether the system to be examined is linear or nonlinear. The differential equation of motion can be conveniently written in operator form as;

$$R(u) = f(t) \qquad (1)$$

where R is an operator. For example, in the case of a system with the equation of motion

$$m\ddot{u} + c\dot{u} + ku = f(t) \tag{2}$$

the operator R is;

$$R = m\frac{d^2}{dt^2} + c\frac{d}{dt} + k \tag{3}$$

Say $R(u_1(t)) = f_1(t)$ and $R(u_2(t)) = f_2(t)$

then a system is linear if

$$R(\alpha u_1(t) + \beta u_2(t)) = \alpha f_1(t) + \beta f_2(t)$$

Example

Test linearity for the differential equation shown

$$\dot{y} + a(t)y = f(t) \tag{4}$$

we have;

$$\begin{aligned} \dot{y}_1 + a(t)y_1 &= f_1(t) \\ \dot{y}_2 + a(t)y_2 &= f_2(t) \end{aligned} \tag{5a,b}$$

summing eqs (5a) and (5b) one gets;

$$(\dot{y}_1 + \dot{y}_2) + a(t)(y_1 + y_2) = f_1(t) + f_2(t) \tag{6}$$

then if we define

$$y_3 = y_1 + y_2 \tag{7}$$

and

$$\mathrm{f}_3 = \mathrm{f}_1 + \mathrm{f}_2 \tag{8}$$

we can write

$$\dot{\mathrm{y}}_3 + \mathrm{a(t)y}_3 = \mathrm{f}_3 \tag{9}$$

which shows the equation is linear. Note that the equation satisfies linearity even though the term a(t) is an explicit function of time.

Example

Test linearity for the differential equation shown

$$\dot{y} + a(t)y^3 = f(t) \tag{10}$$

following the same approach as in the previous example we write;

$$\begin{aligned} \dot{y}_1 + a(t){y_1}^3 &= f_1(t) \\ \dot{y}_2 + a(t){y_2}^3 &= f_2(t) \end{aligned} \tag{11a,b}$$

and

$$(\dot{y}_1 + \dot{y}_2) + a(t)({y_1}^3 + {y_2}^3) = f_1(t) + f_2(t) \tag{12}$$

taking

$$y_1 + y_2 = y_3 \tag{13}$$

and

$$f_1 + f_2 = f_3 \tag{14}$$

one gets

$$\dot{y}_3 + a(t)[y_3{}^3 + something] = f_3$$

so

$$\dot{y}_3 + a(t) y_3{}^3 \neq f_3 \tag{15}$$

and we conclude the system is nonlinear.

Other Terminology Associated with Classification

- When the coefficients of the differential equation depend on time the system is time varying. When they do not depend on time, the system is said to be time invariant.
- Time invariant systems have the desirable characteristic that a shift in the input produces an equal shift in the output, i.e.,

If

$$R(u(t)) = f(t)$$

then

$$R(u(t-\tau)) = f(t-\tau)$$

When a system is linear the response can be built by treating the excitation as the sum of components and invoking the principle of superposition. This possibility simplifies the treatment of complex loading cases. We should note that the assumption of linearity is often reasonable provided the amplitude of the response is sufficiently small.

Sources of nonlinearity:

1. Material Nonlinearity
   One source of nonlinearity that is commonly encountered is that due to nonlinear stress-strain relationships in the materials.

2. Geometric Nonlinearity

   Is encountered when the geometry suffers large changes during the loading. In these problems the equilibrium equations must be written with reference to the deformed configuration.

Dynamic Equilibrium

The equations of motion can be viewed as an expression of energy balance or as a statement of equilibrium. In a dynamic setting equilibrium involves not only applied loads and internal resistance but also inertial and damping forces. The inertia forces are readily established from Newton's Law as the product of the masses times the corresponding accelerations.

In contrast with inertial loads, damping forces derive from a generally undetermined internal mechanism. For analysis, the true damping mechanism is replaced by an idealized mathematical model that is capable of approximating the rate of energy dissipation observed to occur in practice.

Consider the situation where a system is subjected to a certain imposed displacement u(t). Assume that we can monitor the force f(t) which is required to impose the displacement history and that the resistance of the system from inertia and from stiffness is known as a function of the displacement and the acceleration. The resistance that derives from damping can then be computed as

$f_d(t)$ = f(t) - inertia - elastic restoring force.

The work done by the damping force in a differential displacement (which equals the dissipated energy) is given by;

$$dW = f_D(t).du$$
$$dW = f_D(t)\frac{du}{dt}dt \qquad (16a,b,c)$$
$$dW = f_D(t)\dot{u}dt$$

so the amount of energy dissipated between times t_1 and t_2 is;

$$W = \int_{t_1}^{t_2} f_D(t)\,\dot{u}\,dt \tag{17}$$

The most widely used damping model takes the damping force as proportional to the velocity, namely;

$$f_D(t) = c\dot{u}(t) \tag{18}$$

where c is known as the damping constant.

To examine the implication of the assumption of viscosity on the dissipated energy we substitute eq.18 into eq.17 and get;

$$W = c\int_{t_1}^{t_2} \dot{u}^2\,dt \tag{19}$$

Consider the situation where the imposed displacement is harmonic, i.e

$$\begin{aligned} u(t) &= A\sin\Omega t \\ \dot{u}(t) &= A\Omega\cos\Omega t \end{aligned} \tag{20a,b}$$

substituting eqs20 into eq.19 we get

$$\begin{aligned} W &= \int_0^{2\pi/\Omega} cA^2\Omega^2(\cos\Omega t)^2\,dt \\ W &= cA^2\Omega^2\int_0^{2\pi/\Omega}(\cos\Omega t)^2\,dt \end{aligned} \tag{21}$$

taking

$$\Omega t = \Theta, dt = \frac{d\Theta}{\Omega} \tag{22}$$

one gets

$$W = cA^2\Omega^2 \int_0^{2\pi} \frac{(\cos\Theta)^2}{\Omega} d\Theta \tag{23}$$

which gives

$$W = \pi cA^2\Omega \tag{24}$$

We conclude, therefore, that when the damping model is assumed viscous the energy dissipated per cycle is proportional to the frequency and proportional to the square of the amplitude of the imposed displacement.

Test results indicate that the energy dissipated per cycle in actual structures is indeed closely correlated to the square of the amplitude of imposed harmonic motion. Proportionality between dissipated energy and the frequency of the displacements, however, is usually not satisfied. In fact, for a wide class of structures results show that the dissipated energy is not significantly affected by frequency (at least in a certain limited frequency range). The viscous model, therefore, is often not a realistic approximation of the true dissipation mechanism. A model where the dissipated energy W is assumed to be independent of Ω can often provide a closer representation of the true dissipation mechanism than the viscous model. When dissipation is assumed to be frequency independent the damping is known as material, structural or hysteretic.
A question that comes to mind is - how is the damping force $f_d(t)$ related to the response u(t) if the energy dissipation is independent of Ω ? After some mathematical manipulations it can be shown that the damping force is related to the response displacement as;

$$f_{D(t)} = G \cdot \int_{-\infty}^{\infty} \frac{u(\tau)}{t-\tau} d\tau \tag{25}$$

where G is a constant. Noting that the Hilbert transform of a function f(t) is defined by

$$H(f_{(t)}) = \frac{1}{\pi} \cdot \int_{-\infty}^{\infty} \frac{f(\tau)}{t-\tau} d\tau \tag{26}$$

we conclude that the damping force in the hysteretic model is proportional to the Hilbert transform of the displacement.

It is worth noting that the Hilbert transform of a function is a non-causal operation – meaning that the value of the transform at a given time depends not only on the full past but also on the full future of the function being transformed (note that the limits of the integral are from -∞ to ∞). Since physical systems must be causal, the hysteretic assumption can not be an exact representation of the true damping mechanism. Nevertheless, as noted previously, the hystertic idealization has proven as a useful approximation. The equation of motion of a SDOF with hysteretic damping in the time domain is given by;

$$m\ddot{u} + H(u(t)) + k\,u = f(t) \tag{27}$$

Which can only be solved by iterations because the damping force is non-causal. In practice hysteretic damping has been used fundamentally in the frequency domain (where the lack of causality is treated in a much simpler fashion).

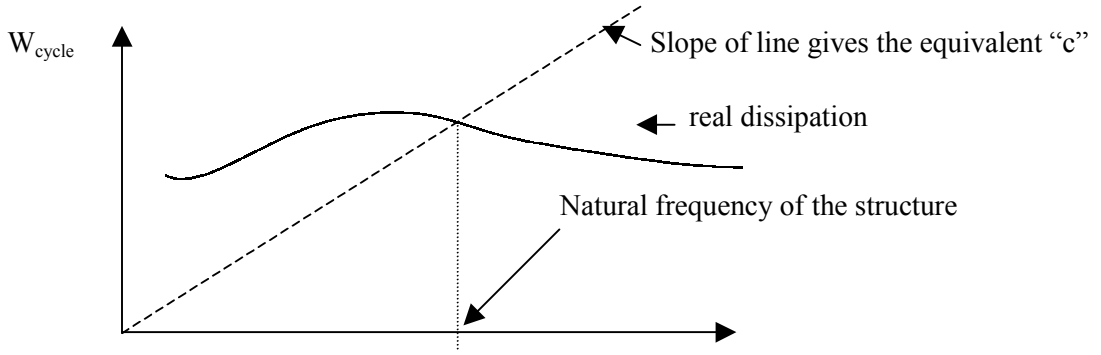
Equivalent Viscous Damping

Although hysteretic damping often provides a closer approximation than the viscous model, the viscous assumption is much more mathematically convenient and is thus widely used in practice. When a structure operates under steady state conditions at a frequency Ω the variation of the damping with frequency is unimportant and we can model the damping as viscous (even if it is not) using:

$$c = \frac{(W_{cycle})_{experimental}}{\delta A^2 \Omega} \tag{28}$$

For transient excitation with many frequencies the usual approximation is to select the damping

constant c such that the energy dissipation in the mathematical model is correct at resonance, i.e when the excitation frequency Ω equals the natural frequency of the structure. This is approach is based on the premise that the vibrations will be dominated by the natural frequency of the structure.



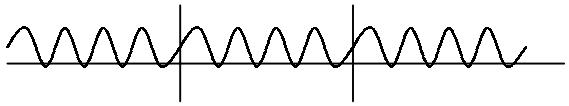
Types of Excitation

- Deterministic - Excitation is known as a function of time

- Random – (stochastic) – excitation is described by its statistics.

A random process is the term used to refer to the underlying mechanism that generates the random results that are perceived.

A sample from a random process is called a realization.

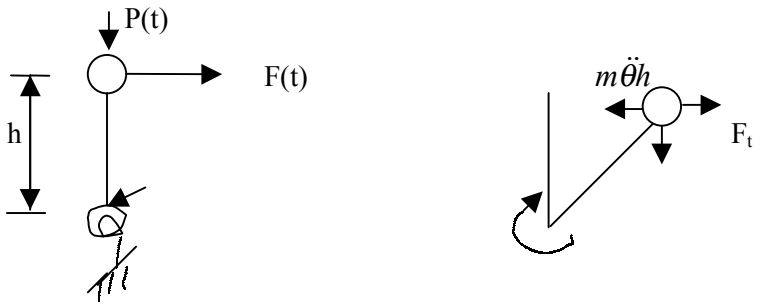


A random process is known as Stationary when the statistics are not a function of the window used to compute them. The process is non-stationary when the statistics depend on time.

The computation of the response of structures to random excitations is a specialized branch of structural dynamics known as Random Vibration Analysis.

- Parametric Excitation

We say an excitation is parametric when it appears as a coefficient in the differential equation of motion. In the example illustrated in the sketch the vertical excitation P(t) proves to be parametric.



Assuming the rotation is small and summing moments about the base we get

$$\sum M = 0 \rightarrow F(t)h + P(t)\theta h - m\ddot{\theta}h^2 - K_r\,\theta = 0$$

$$m\ddot{\theta}h^2 + (K_r - P(t)h)\theta = F(t)h$$

Parametric Excitation

Types Of Problems In Structural Dynamics

The various problems in structural dynamics can be conveniently discussed by referring to the equation of motion in operator form, in particular;

$$R\,(u(t)) = f(t) \qquad (28)$$

- *R* and *f(t)* are known and we wish to compute *u(t)* → Analysis

- *R* and *u(t)* are known – we wish to obtain *f (t)* → Identification of the Excitation
  (used for example in the design of accelerometers)
- *u(t)* and *f(t)* are known and we want *R* → System Identification

In addition to the previous types we may wish to control the response.

If the control is attempted by changing the system properties (without external energy) we have Passive Control. If external energy is used to affect the response the control is known as active. The basic equation in active control is;

$$R\,(u(t)\,) = f(t) \;+\; G\,u(t) \tag{29}$$

where *G* is known as the gain matrix (the objective is to define a G to minimize some norm of u(t)).

Solution of the Differential Equation of Motion for a SDOF System

The equation of motion of a viscously damped SDOF can be written as;

$$\mathrm{M}\,\ddot{\mathrm{U}} + \mathrm{C}\,\dot{\mathrm{U}} + \mathrm{KU} = \mathrm{f(t)} \tag{29}$$

The homogeneous equation is:

$$M\,\ddot{U} + C\,\dot{U} + KU = 0 \tag{30}$$

and the solution has the form:

$$U_h(t) = G\,e^{st} \tag{31}$$

substituting eq.31 into 30 one gets;

$$(Ms^2 + Cs + K)Ge^{st} = 0 \tag{32}$$

therefore, except for the trivial solution corresponding to G = 0 it is necessary that;

$$Ms^2 + Cs + K = 0 \tag{33}$$

dividing by the mass and introducing well known relationships one gets

$$s^2 + \tfrac{C}{M}s + \tfrac{K}{M} = 0 \tag{34}$$

$$\tfrac{K}{M} = \omega^2 \tag{35}$$

$$\tfrac{C}{M} = 2\omega\zeta \tag{36}$$

$$s^2 + (2\omega\zeta)s + \omega^2 s = 0 \tag{37}$$

$$s = -\omega\zeta \pm \sqrt{(\omega\zeta^{\,2} - \omega^2} \tag{38}$$

so the roots are;

$$s_{1,2} = -\omega\zeta \pm \omega\sqrt{\zeta^2 - 1} \tag{39}$$

The homogeneous part of the solution is, therefore;

$$U_{h(t)} = G_1 e^{s1t} + G_2 e^{s2t} \tag{40}$$

The particular solution $U_p(t)$ depends on the function, in general:

$$U_{(t)} = G_1 e^{s1t} + G_2 e^{s2t} + U_p(t) \tag{41}$$

where the constants G_1 & G_2 depend on the initial conditions. In particular, at $t = 0$

$$U(t) = U_0 \tag{42}$$

and

$$\dot{U}(t) = \dot{U}_0 \tag{43}$$

therefore,

$$U_0 = G_1 + G_2 + U_{p(t=0)} \tag{44}$$

and

$$\dot{U}_0 = s_1 G_1 + s_2 G_2 + \dot{U}_{p(t=0)} \tag{45}$$

In matrix form the equations can be written as:

$$\begin{bmatrix} 1 & 1 \\ s_1 & s_2 \end{bmatrix} \begin{Bmatrix} G_1 \\ G_2 \end{Bmatrix} = \begin{Bmatrix} U_0 - U_{p(0)} \\ \dot{U}_0 - \dot{U}_{p(0)} \end{Bmatrix} \tag{46}$$

and, solving for G_1 & G_2 one gets;

$$\begin{Bmatrix} G_1 \\ G_2 \end{Bmatrix} = \frac{1}{s_2 - s_1} \begin{bmatrix} s_2 & -1 \\ -s_1 & 1 \end{bmatrix} \begin{Bmatrix} U_0 - U_{p(0)} \\ \dot{U}_0 - \dot{U}_{p(0)} \end{Bmatrix} \tag{47}$$

<u>Free Vibration</u> $(U_p = 0)$ **:**

From eq.(47) one gets;

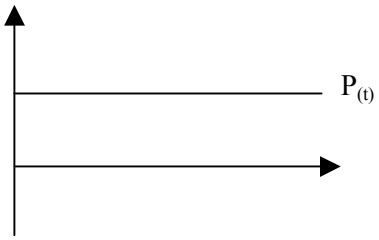
$$G_1 = \frac{s_2 U_0 - s_1 \dot{U}_0}{s_2 - s_1} \tag{48}$$

$$G_2 = \frac{-s_1 U_0 + \dot{U}_0}{s_2 - s_1} \tag{49}$$

so the solution is;

$$U_{(t)} = G_1 e^{s1t} + G_2 e^{s2t} \tag{50}$$

Step Load



Postulate the particular solution

$$U(t) = N \tag{51}$$

Then the equation of motion becomes:

$$M\ddot{U} + C\dot{U} + KU = P \tag{52}$$

substituting the function and its derivatives into the equation one gets;

$$K N = P \tag{53}$$

therefore;

$$N = \frac{P}{K} = \Delta_{st} \tag{54}$$

where Δ_{st} is the static deflection.

G_1 & G_2 are obtained from eq.47 using;

$$U_{p(0)} = \Delta_{st};$$

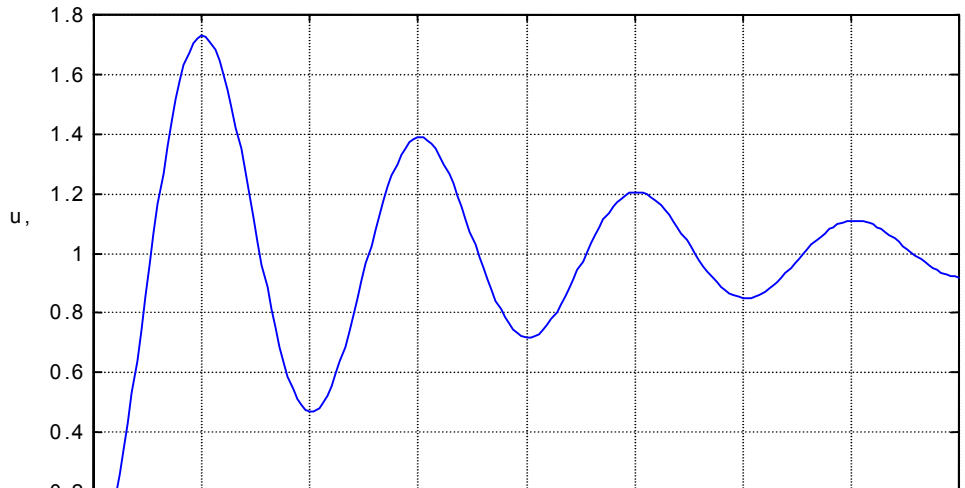
$$\dot{U}_{p(0)} = 0$$

A MATLAB routine that implements the previous equations is presented next.

```
%**********************
%    FVC.M
%**********************
% Program to compute the response of a SDOF system using complex algebra.
% Either free vibration or the vibration for a step load can be obtained.
% Variables
% T = Period
```

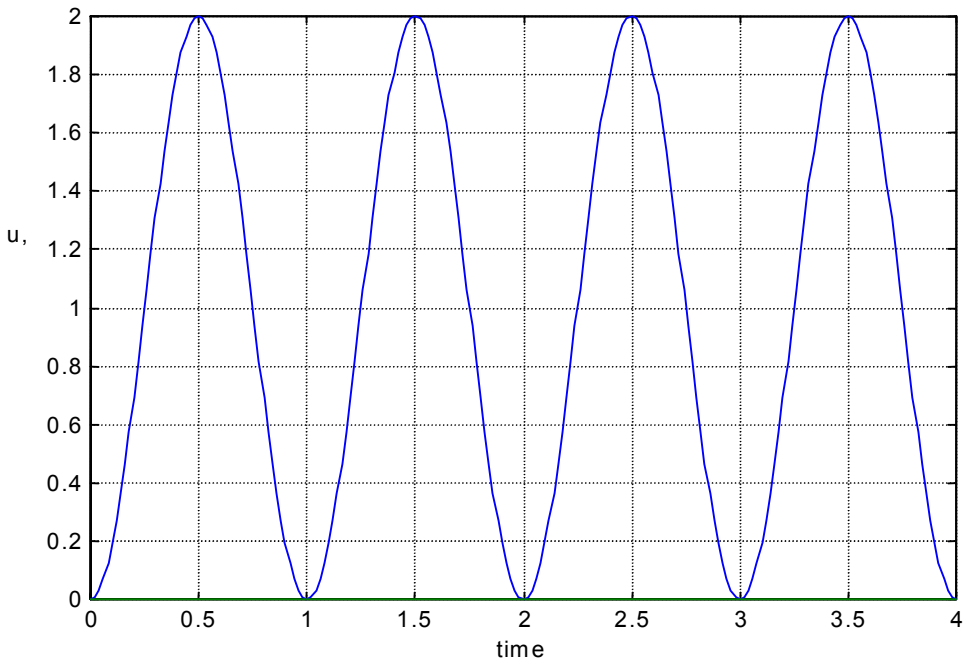
```
% z = Fraction of Critical Damping
% u0 = Initial Displacement
% v0 = Initial Velocity
% ds = Static Deformation due to Step Load
% dt = time step 1/50 of the period to ensure a smooth plot.
T = 1;
z = 0.1;
u0 = 0;
v0 = 0;
dt = T/50;
ds = 1;
w = (2*pi)/T;
t=0:dt:4*T;  %Time range from 0 to 4T with increments of dt
% Evaluate the roots
s1 = -w*z+w*sqrt(z^2-1);
s2 = -w*z-w*sqrt(z^2-1);
A = [s2 -1;   -s1 1];
A = 1/(s2-s1)*A;
G = A*[u0-ds v0]';
u = G(1)*exp(s1*t) + G(2)*exp(s2*t) + ds;
% While u will be real, the computer will calculate an imaginary component that is very small
% due to round-off error.
plot(t,real(u),t,imag(u)); %plots both real and imaginary parts of u
grid;
xlabel('time');
ylabel('u, displacement');
```

FIGURE 1



The solution for a step load where Δ_{st} =1 is illustrated in Figure 1. In this case, the damping is 10% and we see that the peak is approximately equal to 1.78. It can be shown that if the damping is zero, the peak would be 2* Δ_{st} (See Figure 2). Therefore damping is not that efficient in reducing the response for a step load. This is generally true when the maximum response occurs early.

FIGURE 2



Revisit

Eq.39 ;

$$s_{1,2} = -\omega\zeta \pm \omega\sqrt{\zeta^2 - 1}$$

since the damping ratio is typically much less than one it is convenient to write the roots as;

$$s_{1,2} = -\omega\zeta \pm i\omega\sqrt{1-\zeta^2} \tag{55}$$

which highlights the fact that the roots are typically complex conjugates. Designating

$$\omega_D = \omega\sqrt{1-\zeta^2} \tag{56}$$

one can write;

$$s_{1,2} = -\omega\zeta \pm i\omega_D \tag{57}$$

While the roots and the constants G_1 and G_2 are generally complex, the solution u(t) is, of course, always real. This can be easily demonstrated analytically by introducing Euler's identity $e^{i\theta t} = \cos(\theta t) + i\sin(\theta t)$ and carrying out the algebra. Numerical confirmation is also found in the output from the program since the imaginary part of the result is plotted and found to be on top of the x-axis, showing that it is zero.

Computation of the Response by Time Domain Convolution

In the previous section we looked at the computation of the response as a direct solution of the differential equation of motion. In this section we examine an alternative approach where the solution is obtain by convolution. An advantage of the convolution approach is that one obtains a general expression for the response to any applied load.

We begin by defining the Dirac delta or impulse function δ(t) in terms of its basic property, namely

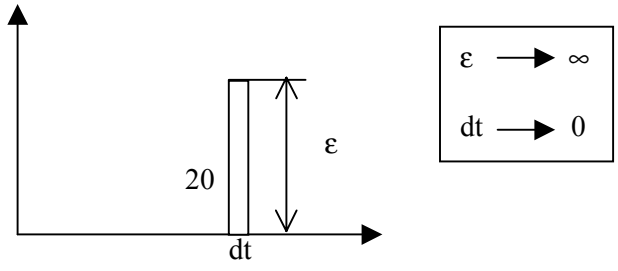
$$\delta(t-a) = 0 \quad \text{for} \quad t \neq a \tag{58}$$

$$\int_{-\infty}^{\infty} \delta(t-a)\, dt = 1 \tag{59}$$

From the previous definition it follows that;

$$U(a) = \int_{-\infty}^{\infty} U(t)\,\delta(\tau - a)\,d\tau \tag{60}$$

A graphical representation of the Dirac Delta function is depicted in the figure.

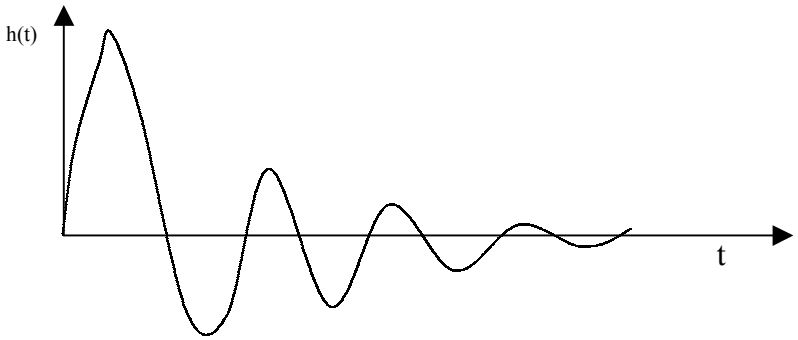


Consider a SDOF system subjected to a Dirac Delta function

$$M\ddot{U} + C\dot{U} + KU = \delta_{(t)}$$

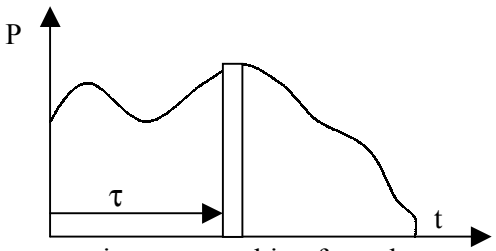
and designate the solution by h(t), namely:

$$U_t = h(t) \qquad \text{(for an impulse at t = 0)}$$

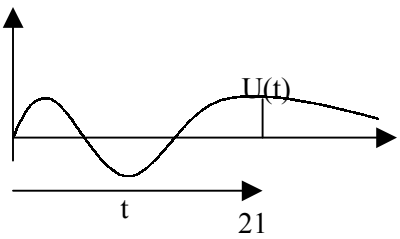


h(t) is known as the impulse response of the system.

Now consider an arbitrary load:



We can think of the response at time t as resulting from the superposition of properly scaled and time shifted impulse responses.



$$U_{(t)} = \int_0^t P(\tau)h(t-\tau)d\tau \tag{61}$$

The integral in eq.61 is known as the convolution integral (also refered as Duhamel's integral in the structural dynamics literature). Note that once h(t) is known, the response to any load P is given by a mathematical operation and does not require that one consider "how" to obtain a particular solution. In summary, for a linear system that starts at rest the response is given by;

$$U_{(t)} = \int_0^t P(\tau)h(t-\tau)d\tau \tag{62}$$

If the system is not at rest at (t = 0), we must add the free vibration due to the initial conditions.

Frequency Domain Analysis

In the previous section we found that the response of a linear system can be computed in the time domain by means of the convolution integral.

$$\text{u(t)} = \int_0^t h(t-\tau)\,p(\tau)d\tau$$

which is also symbolically written as

$$\text{u(t)} = \text{h(t)*p(t)}$$

A disadvantage in the previous approach is that the numerical cost of convolution is high, as we shall see in the development that follows, by transferring the equation to the frequency domain the convolution is rendered a multiplication and significant savings in the numerical effort can result. Before entering into details of the actual frequency domain solution, however, some preliminaries on Fourier Analysis are presented.

Fourier Transform

The Fourier Transform of a function f(t) is defined as;

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt = A(\omega) e^{i\phi(\omega)}$$

as can be seen, the FT is generally a complex function of the real variable ω. It is illustrative then to write it as;

$$F(\omega) = R(\omega) + I(\omega)\, i$$

where, by introducing Euler's identity, one can see that;

$$R = \int_{-\infty}^{\infty} f(t) \cos(\omega t) dt$$

and

$$I = \int_{-\infty}^{\infty} f(t) \sin(\omega t) dt$$

Note that if f(t) is real R(ω) = R(-ω) and I(ω) = - I(-ω).

For F(ω) to exist

$$\int_{-\infty}^{\infty} |f(t)|\, dt \; < \infty$$

<u>Properties of the Fourier Transform</u>

1) Linearity $f1(t) \rightarrow F1(\omega)$

$f2(t) \rightarrow F2(\omega)$

then

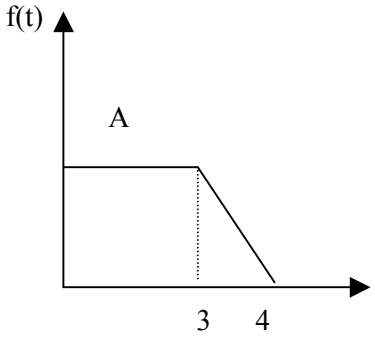
$$a\, f1(t) + b\, f2(t) \rightarrow a\, F1(\omega) + b\, F2(\omega)$$

2) Time Shifting

$$f(t) \rightarrow F(\omega) = A(\omega)\, e^{(i\phi(\omega))}$$

$$f(t-t_0) \rightarrow F(\omega) \exp(-i\omega t_0) = A(\omega)\, e^{(i(\phi(\omega)-\omega t_o))}$$

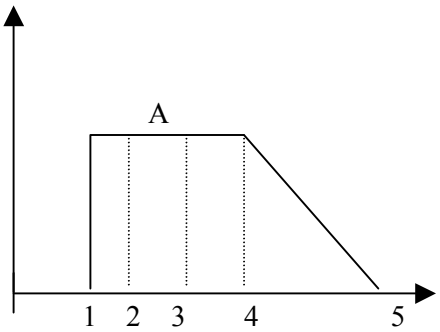
Illustration



t	f(t)	$t-t_0$	$f(t-t_0)$
0	A	-1	0
1	A	0	A
2	A	1	A
3	A	2	A
4	A	3	A

$f(t-t_0)$

$t_0 = 1$



Proof:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i*w*t}dt$$

$$f^*(t) = f(t-t_0)$$

$$F^*(\omega) = \int_{-\infty}^{\infty} f^*(t)e^{-w*t}dt$$

taking $n = t - t_0$

$$t = n + t_0$$

$$dn = dt$$

therefore

$$F^*(\omega) = \int_{-\infty}^{\infty} f^*(t)e^{-i*w*(n+t_0)}dn = e^{-i*w*t_0} \int_{-\infty}^{\infty} f^*(t)e^{-i*w*n}dn$$

$$F^*(\omega) = e^{-i*w*t} \int_{-\infty}^{\infty} f(n)e^{-i*w*n}dn$$

$$F^*(\omega) = e^{-i*w*t_0} \quad F(\omega)$$

The inverse formula is

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(w)e^{i*w*t}dw$$

Proof is not presented but it is interesting to note that it involves using the identity

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i*w*t}dw$$

In summary:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i*w*t}dt$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)e^{i*w*t}dw$$

The Convolution Theorem **(***links time domain to frequency domain)*

$$u(t) = \int_{0}^{t} P(\tau)h(t-\tau)d\tau$$

Provided both P(t) and h(t) are causal the expression can be written as;

$$u(t) = \int_{-\infty}^{\infty} P(\tau)h(t-\tau)d\tau$$

say u(t) $\rightarrow$ u(iω)

$$u(i\omega) = \int_{-\infty}^{\infty} u(t)e^{-w*t}dt = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} P(\tau)h(t-\tau)d\tau\right]e^{-w*t}dt$$

reversing the order of integration we get;

$$u(i\omega) = \int_{-\infty}^{\infty}\left[\int_{\infty}^{\infty} P(\hat{o}(\hat{o})-h\hat{o})\,\bar{e}^{w*t}dt\right]dt$$

or

$$u(i\omega) = \int_{-\infty}^{\infty} P(\hat{o}\left[\int_{-\infty}^{\infty} h(t-\hat{o})\,\bar{e}^{w*t}dt\right]d\hat{o}$$

now taking

$$t - \tau = n$$

we have

$$dt = dn$$

and

$$t = n + \tau$$

substituting one gets

$$u(i\omega) = \int_{-\infty}^{\infty} P(\hat{o}\left[\int_{-\infty}^{\infty} h(n)e^{-i*w*(n+\hat{o})}dn\right]d\hat{o}$$

$$u(i\omega) = \int_{-\infty}^{\infty} P(\hat{o}\,(\eth^{i*w*\hat{o}} \left[\int_{-\infty}^{\infty} h(n) e^{-i*w*n} dn \right] d\hat{o}$$

$$u(i\omega) = \int_{-\infty}^{\infty} P(\hat{o}\,(\eth^{i*w*\hat{o}} h(iw) d\hat{o} \quad u(iw) = h(iw) \int_{-\infty}^{\infty} P(\hat{o}\,(\eth^{i*w*\hat{o}} d\hat{o}$$

or

$$u(i\omega) = h(i\omega)\ P(i\omega)$$

Numerical cost:

$$u(t) = \int_0^t P(\tau) h(t-\tau) d\tau$$ say this requires Z units of cpu time

In the frequency domain the following operations are required;

P(t) = P(iw) $\rightarrow$ x units

h(t) = h(iw) $\rightarrow$ y units

(multiplication) P(iw) h(iw) $\rightarrow \approx$ 0 (very small)

Taking the inverse F.T $\rightarrow$ g units

Total cpu in the frequency domain approach $\approx$ x+y+g. While this total used to be larger than Z, the development of the Fast Fourier Transform algorithm has changed this result making the frequency domain solution the more efficient approach in many cases.

Property of differentiation of Fourier Transformation:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-w*t} dt$$

To integrate by parts, we define

$$f(t) = u \rightarrow f'(t)\, dt = du$$

therefore

$$e^{-i*w*t}\,dt = dv \rightarrow v = \frac{e^{-i*w*t}}{-iw}$$

and we get

$$\int_{-\infty}^{\infty} f(t)e^{-i*w*t}dt = \frac{f(t)e^{-i*\omega*t}}{-i\omega} - \int_{-\infty}^{\infty}\frac{f'(t)e^{-i*w*t}}{iù}dt$$

$$= \int_{-\infty}^{\infty}\frac{f'(t)e^{-i*w*t}}{iw}dt = \frac{1}{wi}\int_{-\infty}^{\infty} f'(t)e^{-i*w*t}dt$$

or

$$i\omega \int_{-\infty}^{\infty} f(t)e^{-i*w*t}dt = \int_{-\infty}^{\infty} f'(t)e^{-i*w*t}dt$$

We conclude that the Fourier Transform of $\frac{d}{dt}$ f(t) is simply (iω)* the Fourier transformation of f(t), namely;

$$F\left(\frac{d}{dt}f(t)\right) = i\omega\, F(\omega)$$

This can be easily extended to state that,

$$F\left(\frac{d^n}{dt^n}f(t)\right) = (i\omega)^n F(\omega)$$

Consider now the equation of motion of a SDOF system,

$$m\ddot{u} + c\dot{u} + ku = P(t)$$

taking the Fourier transform of both sides one gets;

$$m\,(i\omega)^2\,u(i\omega) + c(i\omega)u(i\omega) + ku(i\omega) = P(i\omega)$$

$$u(i\omega)\,(-\omega^2 m + k + c\omega i) = P(i\omega)$$

$$u(i\omega) = \frac{P(iw)}{-w^2 m + k + cwi}$$

and since

$$u(i\omega) = P(i\omega)\, h(i\omega)$$

we conclude that

$$h(i\omega) = \frac{1}{-w^2 m + k + cwi}$$

we can establish, therefore, that

$$\int_{-\infty}^{\infty} \frac{1}{m\, wd} e^{-i*wd*t} \sin(wd\, t)\, e^{-i*w*t} dt = \frac{1}{(-w^2 m + k) + cwi}$$

Fourier transformation of Dirac delta function

$$F(\delta(t)) = \int_{-\infty}^{\infty} \delta(t) e^{-iwt} dt = 1$$

We have previously noted that the response to a Dirac Delta function is the impulse response of the system, it is easy to show that the Fourier transform of the impulse response equals the frequency response function. Consider a SDOF subjected to a Dirac Delta function;

$$m\ddot{h}(t) + c\dot{h}(t) + k\,h(t) = \delta(t)$$

where the notation h(t) has been introduced to emphasize that the computed response is the impulse response function. Taking a Fourier transform we get;

$$h(i\omega) = \frac{1}{[(-\omega^2 m + k) + c\omega i]}$$

which is the expected result.

Mathematical Formulation of Frequency Domain Analysis

Say the equations of motion for a MDOF system are:

$$[M]\{\ddot{u}\}+[c]\{\dot{u}\}+[K]\{u\}=\{f(t)\}$$

and for notational convenience we do not use the brackets from here on. This equation can be transformed to the frequency domain as

$$M(i\omega)^2\ u(i\omega) + c(i\omega)\ u(i\omega) + K\ u(i\omega) = f(i\omega)$$

or

$$[(-\omega^2 M + K) + c\,\omega i]\ u(i\omega) = f(i\omega)$$

therefore

$$u(i\omega) = [(-\omega^2 M + K) + c\,\omega i]^{-1}\ f(i\omega)$$

so

$$u(t) = \frac{1}{2}\int_{-\infty}^{\infty} u(i\omega)e^{i\omega t}\,d\omega$$

or

$$u(t) = \text{IFT}\ u(i\omega)$$

A special case often found in practice is that where f(t) = g z(t), where g is a fixed spatial distribution. In this case one gets

$$u(i\omega) = [-\omega^2 M+K+c\omega i]^{-1}\ \{g\}\ z(t)$$

Numerical Implementation of Frequency Domain Analysis

In practice we usually can not obtain the FT analytically and we resort to a numerical evaluation.

If f(t) is discretized at spacing Δt, then the highest frequency that can be computed in the numerical transform is the **Nyquist frequency**:

$$W_{NYQ} = \pi/\Delta t$$

If the duration of f(t) is "t_{max}", then the frequency spacing of the discrete FT is

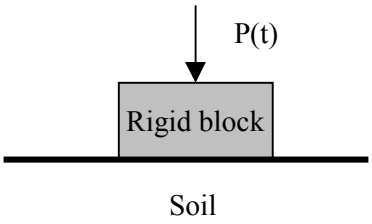
$$d\omega = 2\pi / t_{max}.$$

If you have the FT analytically and you are computing the IFT numerically, then the time function you get, will repeat every $t_p = 2\pi / d\omega$ where dω is the frequency step used in the evaluation.

Reasons for using a frequency domain analysis

- It may be more convenient than a solution in the time domain
- The mathematical model has frequency dependent terms.

Models with frequency dependent terms are often encountered when describing the properties of a dynamic system in terms of a reduced set of coordinates. Consider for example the case of a rigid block on soil subjected to a vertical dynamic load P(t).



If we assume the soil as linear then the impulse response may be obtained (albeit approximately) by subjecting the area where the block rests to a load that grows and decays very quickly (through a rigid lightweight plate). If we take the Fourier Transform of the impulse response we obtain a frequency response function which can be expressed as;

$$h_e(i\omega) = \frac{1}{m_e(i\omega)^{\wedge}2 + k_e + c_e\, i\omega}$$

If the soil medium where truly a SDOF viscously damped system we would find that there are values of m_e , k_e and c_e for which the previous equation matches the results obtained. However, since the response of the medium is much more complex than that of a SDOF we find that it is

not possible to match the experimentally obtained function with the three free parameters m_e , k_e and c_e. While we can try to obtain a "best fit" solution it is also possible to treat m_e , k_e and c_e as frequency dependent parameters. In this case we can select values for these coefficients to match the experimental curve at each frequency. Since at each frequency the actual value of the frequency response is a complex number (two entries) we only need two free parameters and it is customary to take $m_e = 0$. Note, therefore, that we have arrived at a model that "looks" like a SDOF system but with parameters that can only be specified in the frequency domain.

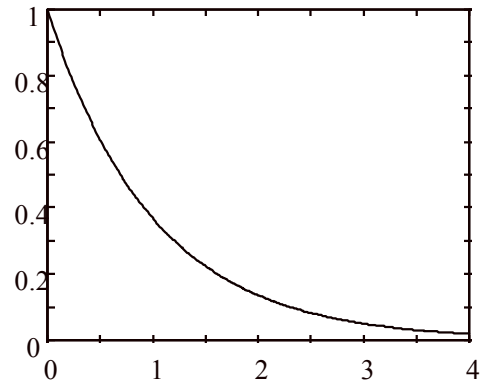
Another way to say the same thing is that in the frequency domain we separate the load into its harmonic components and we can calculate the response to each component using properties that change – the full response is, of course, obtained by superposition.

```
% **********
% OFT.M
% **********
%  Routine to clarify some issues associated with the use of the FFT algorithm in Freq. Domain
% Analysis.
% ***********************************
% The program considers the time function

%                y(t)=U(t) exp(-a*t) which has the FT y(iw)= (a - iw)/(a^2+w^2)

% and can be used to examine how the exact transform and the  FFT compare or to test how the
% IFFT of the true transform compares with the real function.
a=1;
dur=4;
dt=0.01;
t=0:dt:dur;
y=exp(-a*t);
wmax=pi/dt;
dw=2*pi/dur;
% The exact transform is
% yiw=a-i*w/(a^2+w^2);
% Compare the exact transform with the FFT
y1=fft(y*dt);
y1m=fftshift(y1);
ome=-wmax:dw:wmax;
yiw=(a-ome*i)./(a^2+ome.^2);
plot(ome,real(y1m),ome,real(yiw),'g');
pause
% Now we use the exact transform and go back to time.
om1=0:dw:wmax;
om2=-wmax:dw:-dw;
om=[om1 om2];
yiw=(a-om*i)./(a^2+om.^2);
yt=real(ifft(yiw)/dt);
plot(t,yt,t,y,'g');
```

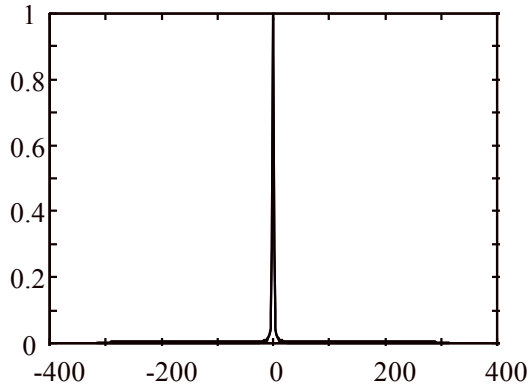
% Note that when you take the FT of a time function you must multiply by dt.
% Note that when you take the IFT of a frequency function the result must be divided by dt.
% Note that in the IFT of this function there is "vibration" around the origin. This is a result of
% Gibbs phenomenon, which appears when the function y(t) is discontinuous. Note that our
% function is discontinuous at the origin.
% The FFT of a time function will be near the real transform if:

% 1) The function is negligible for times after the duration considered.
% 2) The time step dt is such that the values of the true transform for w> wmax are
% negligible - where wmax= pi/dt.

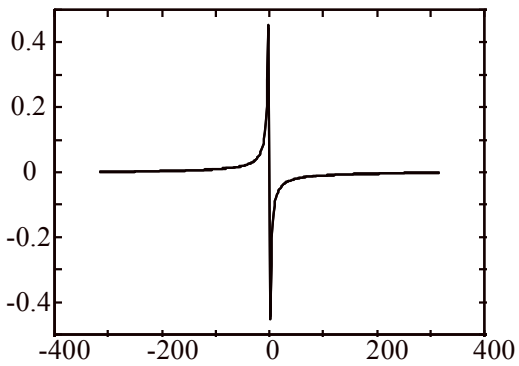
% In any case, the FFT of a discrete time function gives coefficients of a periodic expansion of
% the function.
% The IFFT of a frequency function is periodic with period T=2*pi/dw, the first cycle will be
% close to the real time function if the part of the function beyond the truncation is negligible.



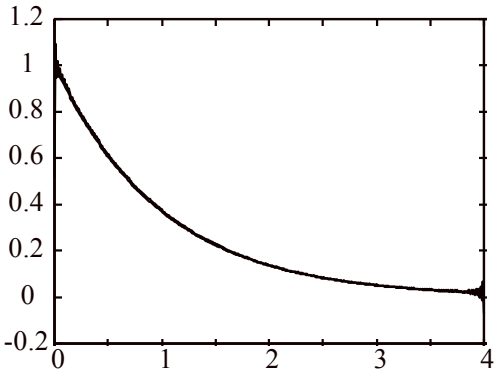
(a)



(b)



(c)



(d)

(a) Time function; (b) Comparison of real parts of FFT and Fourier Transform; (c) Comparison of imaginary parts of FFT and Fourier Transform; (d) Comparison of IFFT and the function itself.

Observations:

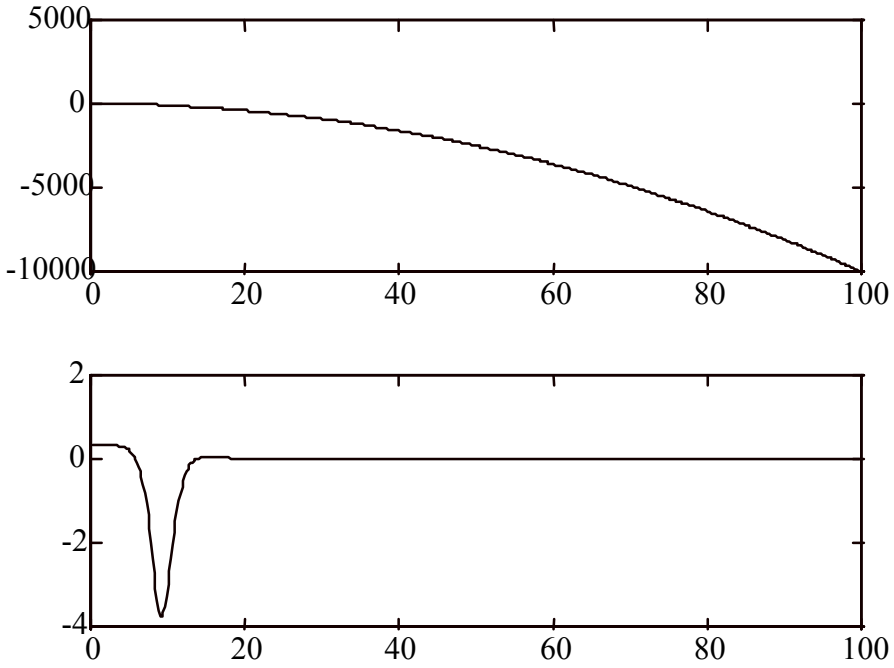
1. In the case shown here the true transform and the FFT are very close because the true Fourier Transform is negligible for frequencies > $\omega_{Nyquist}$.
2. The IFFT gives a very good approximation of the function because $d\omega$ is small enough and $\omega_{Nyquist}$ is large enough.

You can use the program provided to test what happens when you change the parameters.

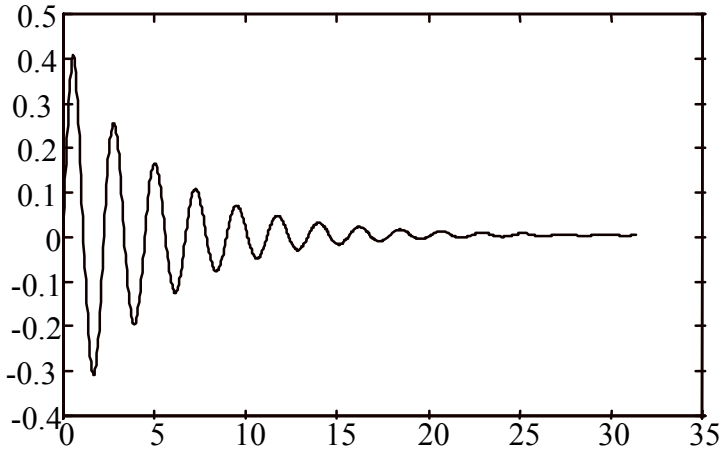
```
% **********
% EFDS.M
% **********

clear

% Program to illustrate how a MDOF can be arbitrarily condensed when the analysis is carried
% out in the frequency domain. In the limiting case, if one is intersted in the response at a single
% coordinate, the condensed system takes the form of a SDOF with frequency dependent
% parameters.

% Theory
% ********************************************************************
% tndof = total number of degrees of freedom
% rdof = reduced number of degrees of freedom to keep
% ntv = number of loads having a different time variation

% u(iw) = (A'*h(iw)*g)*z(iw);
% where:
% A = matrix (rdof , tdof). In each row is full of zeros with a one at each one  of the DOF of
% interest. For example, if we have a 4dof system and we wanted to retain 1 and 4
% A =[1 0 0 0;0 0 0 1];

% h(iw) = [-w^2*m+k+cwi]^-1
% g = spatial distribution matrix (tndof , ntv)
% z(iw) = fourier transform of temporal variation of the loads (ntv , 1)

% ************************
% Special case where rdof = 1 *
% ************************

% The equation for u(iw) can also be written as
% u(iw) = he(iw)*z(iw);
% from where it is evident that he(iw) plays the role of h(iw) in the
% case of a SDOF system.
% ************
% Since he(iw)=(A'*h(iw)*g)=a+bi and h(iw) = 1/((k-mw^2)+cwi)
% we can choose m=0 and solve for k and c in terms of a and b
% the results are k = a/(a^2+b^2) and c = -b/(w*(a^2+b^2))
% **************
```

```
% The first part of the program computes the equivalent parameters for the roof response of a 3
% story shear building having the parameters given next.


% ****** Data ******************
m = eye(3);
k=[40 -40 0;-40 80 -40;0 -40 80];
c=0.05*k;
A=[1 0 0]';
g=[1 1 1]';
dw=0.2;
n=500;
wmax=(n-1)*dw;
% ******************************
i=sqrt(-1);
for j=1:n;
w=(j-1)*dw;
he=A'*inv(k-m*w^2+c*w*i)*g;
hee(j)=he;
a=real(he);
b=imag(he);
d=he*conj(he);
ke(j)=a/d;
ce(j)=-b/(w*d);
end;
omeg=0:dw:(n-1)*dw;
subplot(2,1,1),plot(omeg,ke,'w');
subplot(2,1,2),plot(omeg,ce,'c');
pause
% ************************************************************
% In this part we take he to the time domain to obtain
% the impulse response.
% The trick used in taking the IFFT can be argued based
% on the causality of the impulse response
heec=hee*0;
heec(1)=[ ];
hee=[hee heec];
dt=pi/wmax;
ht=2*real(IFFT(hee))/dt;
t=0:dt:2*pi/dw;
figure
plot(t,ht);
pause
% ****************************************************

% In this part the program computes the response of the roof of the building to a temporal
% distribution z(t)=U(t)*100*sin(2*pi*t)*exp(-0.5*t);
zt=100*sin(2*pi*t).*exp(-0.5*t)
figure
plot(t,zt,'r');
pause
```

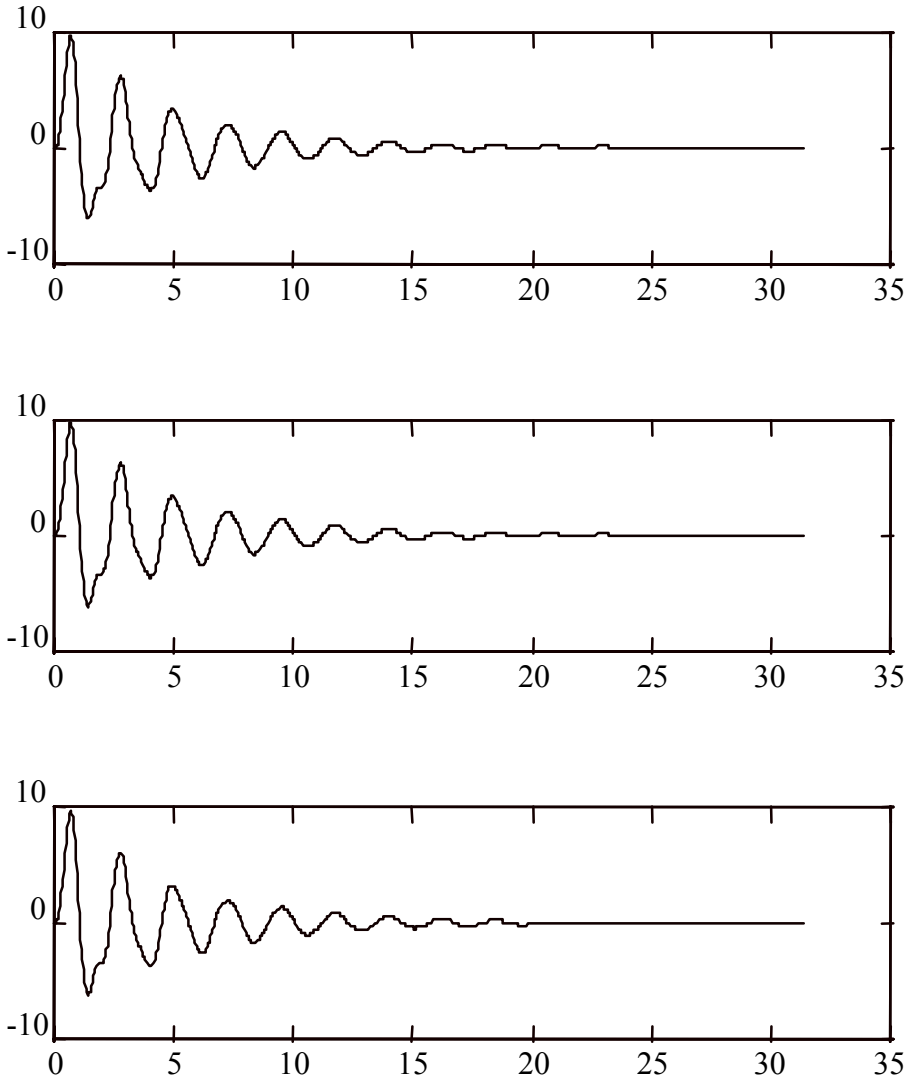
```
% Compute the response in time by convolution;
u1=conv(ht,zt)*dt;
r=length(t);
u1=u1(1:r);

% Calculate in frequency
u2iw=2*(fft(zt*dt).*hee);
u2=real(ifft(u2iw))/dt;

% Check using numerical integration
[tt,dd]=caa(m,c,k,dt,g,zt,g*0,g*0,(r-1));
figure
plot(t,u1);
pause
plot(t,u2,'g');
pause
plot(tt,dd(:,1),'m');
pause

% Show all together
subplot(3,1,1),plot(t,u1);
subplot(3,1,2),plot(t,u2,'g');
subplot(3,1,3),plot(tt,dd(:,1),'m');
```



Equivalent Parameters for the roof response (k and c)



Impulse response for roof respons, h(t)



top = response computed with convolution

center = response computed in frequency

low = response computed with numerical integration (CAA)

Numerical integration

So far we have examined the basics of time domain (convolution) and frequency domain (Fourier Transform) solutions. While in actual practice we may have to implement these techniques numerically, the expressions that we operate with are exact and the error derives exclusively from the numerical implementation. A severe limitation of both convolution and frequency domain analysis, however, is the fact that their applicability is restricted to linear systems. An alternative

approach that is not limited to linear systems is direct numerical integration. The basic idea of numerical integration is that of dividing time into segments and advancing the solution by extrapolation. Depending on how one sets out to do the extrapolation, many different techniques result. An essential distinction between numerical integration methods and the numerical implementation of either convolution or frequency domain techniques is that in the case of numerical integration we don't start with exact expressions but with a scheme to approximate the solution over a relatively short period of time. Of course, in all cases we require that the exact solution be approached as the time step size approaches zero.

In setting up an advancing strategy one can follow one of two basic alternatives. The first one is to formulate the problem so the equation of motion is satisfied at a discrete number of time stations. The second alternative is to treat the problem in terms of weighted residuals so the equations are satisfied "in the average" over the extrapolation segment.

Some Important Concepts Associated with Numerical Integration Techniques

Accuracy:

Accuracy loosely refers to how "good" are the answers for a given time step Δt. A method is said to be of the order *n* ($O(\Delta t)^n$) if the error in the solution decreases with the nth power of the time step. For example, a method is said to be second order if the error decreases by a factor of (at least) four when Δt is cut in half. Generally, we are looking for methods which are at least $O\Delta t^2$.

While it is customary to take Δt as a constant throughout the solution, this is not necessary nor is it always done. For example if we have to model a problem that involves the closure of a gap the stiffness of the system may be much higher when the gap is closed than when its open and it may be appropriate to adjust the time step accordingly. Also, to accurately capture *when* the gap closes it may be necessary to subdivide the step when the event is detected.

Stability:

We say that a numerical integration method is **unconditionally stable** if the solution obtained for undamped free vibration doesn't grow faster than linearly with time, independently of the time step size Δt. Note that linear growth is permitted within the definition of stability. On the other hand, we say that a method is **conditionally stable** if the time step size has to be less than a certain limit for the condition of stability to hold.

Some Classification Terminology for Numerical Integration Methods

Single Step:

A method is known as single step if the solution at t+Δt is computed from the solution at time *t* plus the loading from t to Δt.

Multi step:

A method is multi-step when the solution at t+Δt depends not only on the solution at time t and the loading from t to Δt but also on the solution at steps before time t.

Comment: As one would expect, multi-step methods are generally more accurate than single-step techniques. Multi-step methods, however, typically require special treatment for starting and are difficult to use when the time step size is to be adjusted during the solution interval.

Explicit:

A method is explicit when extrapolation is done by enforcing equilibrium at time t. It is possible to show that in explicit methods the matrix that needs to be inverted is a function of the mass and the damping. Needless to say, when the mass and the damping can be idealized as diagonal the inversion is trivial.

Implicit:

A method is implicit when the extrapolation from t to t+Δt is done using equilibrium at time t+Δt.

Comments: Explicit methods are simpler but are less accurate than implicit ones. In addition, the stability limit of explicit methods is generally much more restrictive than that of implicit methods of similar accuracy. There are no unconditionally stable explicit methods.

Stability Analysis

Let's examine the stability of a rather general class of single step algorithms.

Assume an algorithm can be cast as:

$$\{\hat{X}_{t+\Delta t}\} = [A]\{\hat{X}_t\} + \{L\}P(t+\nu)$$

$\{L\}$ = spatial distribution of loading

$\hat{X}$ = vector of response quantities

$[A]$ = integration approximator

Look at the sequence

$$\hat{X}_1 = A\hat{X}_0 + LP_1 \qquad (\text{say } \nu = \Delta t)$$

$$\hat{X}_2 = A\hat{X}_1 + LP_2 = A(A\hat{X}_0 + LP_1) + LP_2$$

$$= A^2\hat{X}_0 + ALP_1 + LP_2$$

$$\hat{X}_3 = A\hat{X}_2 + LP_3 = A(A^2\hat{X}_0 + ALP_1 + LP_2) + LP_3$$

$$= A^3\hat{X}_0 + A^2LP_1 + AP_2 + LP_3$$

it then follows that in general

$$\hat{X}_n = A^n\hat{X}_0 + \sum_{k=0}^{n-1} A^k LP_{n-k}$$

Stability requires that the maximum eigenvalue in the matrix A have a modulus < 1 (in the scalar case it means that A<1). To explore the stability requirement when [A] is a matrix we first perform a spectral decomposition of [A],

$$[A] = [P][\lambda][P]^{-1}$$

the above is known as the Jordan form of [A] and the operation is known as similarity transformation. In the previous expression [λ] is a diagonal matrix listing the eigenvalues of [A]. A convenient expression for the exponential of [A] can now be obtained by noting that;

$$A^2 = A \cdot A = P\lambda P^{-1}P\lambda P^{-1} = P\lambda\lambda P^{-1} = P\lambda^2 P^{-1}$$

$$A^3 = A^2 \cdot A = P\lambda^2 P^{-1}P\lambda P^{-1} = P\lambda^2\lambda P^{-1} = P\lambda^3 P^{-1}$$

therefore

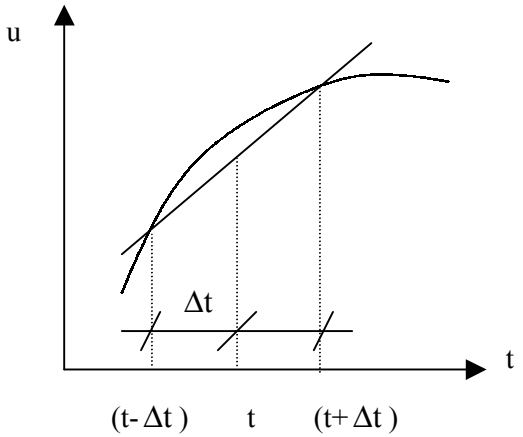
$$A^n = P\lambda^n P^{-1}$$

It is evident, therefore, that to keep A^n from becoming unbounded it is necessary that the largest $|\lambda| < 1$. The largest eigenvalue is called the spectral radius of the matrix, namely;

$$\rho = \max|\lambda|$$

A numerical integration method is unconditionally stable if $\rho < 1$ independently of the time step. A technique is conditionally stable if $\rho > 1$ for $\Delta t >$"SL"and SL is known as the stability limit.

The Central Difference Algorithm:



$$\dot{u}_t = \frac{(u_{t+\Delta t} - u_{t-\Delta t})}{2\Delta t} \tag{1}$$

Consider a Taylor series for the displacement at $t + \Delta t$ (expanded about u(t)):

$$u_{t+\Delta t} = u_t + \dot{u}_t \Delta t + \ddot{u}_t \frac{\Delta t^2}{2} + ...$$

inserting into eq.1 one gets;

$$\ddot{u}_t = \frac{1}{\Delta t^2}[u_{t+\Delta t} - 2u_t + u_{t-\Delta t}] \tag{2}$$

Note that the above equation does not require information further than one time step away from time t. Equilibrium at time t gives:

$$m\ddot{u}_t + c\dot{u}_t + ku_t = P(t) \tag{3}$$

Substituting eqs.1 and 2 into eq.3 and solving for $u_{t+\Delta t}$ one gets after some simple algebra:

$$u_{t+\Delta t} = au_t - bu_{t-\Delta t} + \bar{c}(\frac{P_t}{m}) \ ,$$

where

$$a = \frac{\left(\frac{2m}{\Delta t^2} - k\right)}{\left(\frac{m}{\Delta t^2} + \frac{c}{2\Delta t}\right)}$$

$$b = \frac{\left(\frac{m}{\Delta t^2} - \frac{c}{2\Delta t}\right)}{\left(\frac{m}{\Delta t^2} + \frac{c}{2\Delta t}\right)}$$

$$\bar{c} = \frac{m}{\left(\frac{m}{\Delta t^2} + \frac{c}{2\Delta t}\right)}$$

It can be shown that for a SDOF system (or the *jth* mode of a MDOF system):

$$a = \frac{2 - (\omega\Delta t)^2}{1 + \omega\xi\Delta t}$$

$$b = \frac{1 - \omega\xi\Delta t}{1 + \omega\xi\Delta t}$$

$$\bar{c} = \frac{\Delta t^2}{1 + \omega\xi\Delta t}$$

Define:

$$\hat{X}_t = \begin{Bmatrix} u_t \\ u_{t-\Delta t} \end{Bmatrix}$$

$$\hat{X}_{t+\Delta t} = \begin{Bmatrix} u_{t+\Delta t} \\ u_t \end{Bmatrix}$$

then

$$\begin{Bmatrix} u_{t+\Delta t} \\ u_t \end{Bmatrix} = \begin{bmatrix} a & -b \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} u_t \\ u_{t-\Delta t} \end{Bmatrix} + \begin{Bmatrix} \bar{c} \\ 0 \end{Bmatrix} \frac{P_t}{m}$$

Let's examine stability:

$$\begin{bmatrix} a & -b \\ 1 & 0 \end{bmatrix} \longrightarrow$$ To compute the eigenvalues we need to solve:

$$\begin{vmatrix} a-\lambda & -b \\ 1 & -\lambda \end{vmatrix} = 0$$

$$(a-\lambda)(-\lambda) + b = 0$$

$$-a\lambda + \lambda^2 + b = 0$$

$$\lambda = \frac{a}{2} \pm \sqrt{(a/2)^2 - b}$$

Assume stability limit $\lambda = -1$

$$-1 - \frac{a}{2} = \sqrt{(a/2)^2 - b}$$

$$\left(1 + \frac{a}{2}\right)^2 = \left(\frac{a}{2}\right)^2 - b$$

$$1 + a + \frac{a^2}{4} = \frac{a^2}{4} - b$$

$$1 + a = -b$$

enter the values for a and b:

$$1 + \frac{2 - (\omega\Delta t)^2}{1 + \omega\xi\Delta t} = -\frac{1 - \omega\xi\Delta t}{1 + \omega\xi\Delta t}$$

$$1 + \omega\xi\Delta t + 2 - (\omega\Delta t)^2 = \omega\xi\Delta t - 1$$

$$4 = (\omega\Delta t)^2$$

$$2 = \omega\Delta t$$

$$2 = \frac{2\pi}{T}\Delta t$$

the method, therefore, is conditionally stable and the stability limit is;

$$\Delta t \le \frac{T}{\pi}$$

For the central difference method we need a special starting technique. To derive we consider the equation that predicts the displacements forward, namely

$$u_{t+\Delta t} = a u_t - b u_{t-\Delta t} + \bar{c}\frac{P_t}{m}$$

then, from eqs.1 and 2 one gets:

$$\ddot{u}_0 = \frac{1}{\Delta t^2}\left(u_1 - 2u_0 + u_{-1}\right)$$

$$\dot{u}_0 = \frac{1}{2\Delta t}\left(u_1 - u_{-1}\right) \Rightarrow u_1 = 2\Delta t\dot{u}_0 + u_{-1}$$

then

$$\ddot{u}_0 = \frac{1}{\Delta t^2}\left(2\Delta t\dot{u}_0 + u_{-1} - 2u_0 + u_{-1}\right)$$

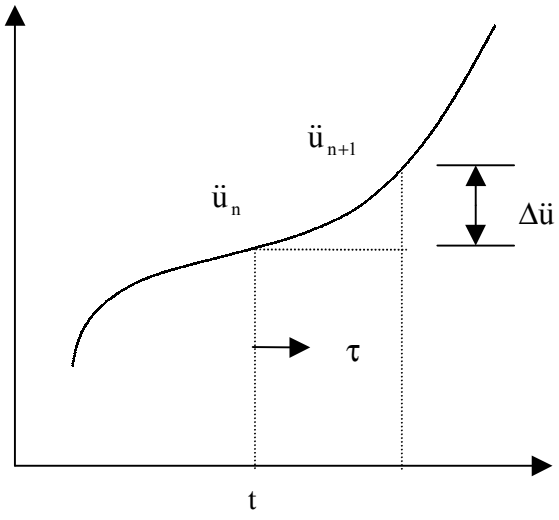
Therefore,

$$u_{-1} = u_0 - \Delta t\dot{u}_0 + \frac{\Delta t^2}{2}\ddot{u}_0$$

which is the desired result.

Newmark's Method

A family of methods known as the Newmark-β method are based on assuming the form of the acceleration within the time step.



The idea in the approach is evident in the equations presented next;

$$\Delta\ddot{u} = \ddot{u}_{n+1} - \ddot{u}_n$$

$$\ddot{u}(\tau) = \ddot{u}_n + \Delta\ddot{u}\alpha(\tau)$$

$$\alpha(\tau) = 0 \quad \text{at} \quad \tau = 0$$

$$\alpha(\tau) = 1 \quad \text{at} \quad \tau = \Delta t$$

Integrating the acceleration one gets

$$\dot{u}(\tau) = \ddot{u}_n \tau + \Delta\ddot{u}\int_0^\tau \alpha(\bar{\tau})d\bar{\tau} + \dot{u}_0$$

for notational convenience we designate

$$\int_0^\tau \alpha(\bar{\tau})d\bar{\tau} = \delta(\tau)$$

Integrating again one gets

$$u(\tau) = \frac{\ddot{u}_n \tau^2}{2} + \Delta\ddot{u}\int_0^\tau \delta(\bar{\tau})d\bar{\tau} + \dot{u}_0\tau + u_0$$

and we now define

$$\int_0^\tau \delta(\bar{\tau})d\bar{\tau} = n(\tau)$$

Evaluating at $\tau = \Delta t$

$$\dot{u}_{n+1} = \ddot{u}_n \Delta t + \Delta\ddot{u}\delta(\Delta t) + \dot{u}_0$$

$$u_{n+1} = \ddot{u}_n \frac{\Delta t^2}{2} + \Delta\ddot{u} \cdot n(\Delta t) + \dot{u}_0 \Delta t + u_0$$

which can be also written as:

$$\dot{u}_{n+1} = \dot{u}_n + [(1-\gamma)\ddot{u}_n + \gamma\ddot{u}_{n+1}]\Delta t$$

$$u_{n+1} = u_n + \dot{u}_n \Delta t + [(\tfrac{1}{2} - \beta)\ddot{u}_n + \beta\ddot{u}_{n+1}]\Delta t^2$$

where γ and β depend on $\alpha(\tau)$ and are easily related to $\delta(\Delta t)$ and $n(\Delta t)$. Specific values of γ and β define specific members of the Newmark Family.

Stability of the newmark algorithm

Results show that for the spectral radius to be <1

$$\frac{0.5-\gamma}{1+\beta(\varpi\Delta t)^2}<0$$

and to ensure that the eigenvalues are complex

$$\beta=\frac{1}{4}(\frac{1}{2}+\gamma)^2-\frac{1}{(\omega\Delta t)^2}$$

Unconditional stability is realized when;

$$\gamma\geq\frac{1}{2}$$

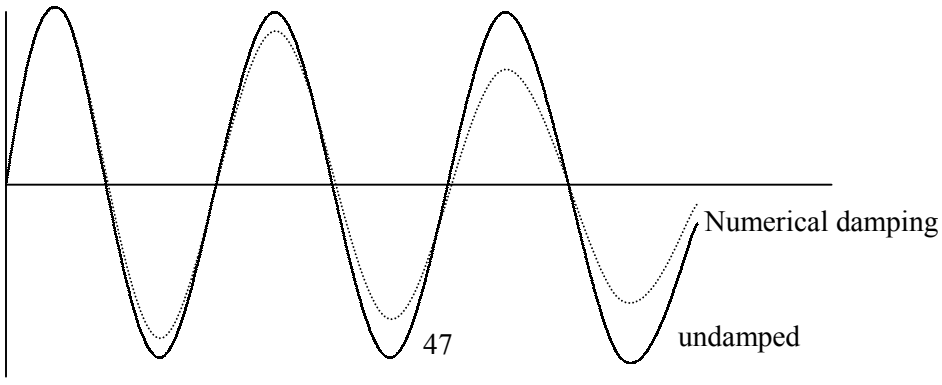
and

$$\beta\geq\frac{1}{4}(0.5+\gamma)^2$$

Members	γ	β	Stability Condition
CAA	1/2	1/4	Unconditionally Stable
LAA	1/2	1/6	$\frac{\Delta t}{T}\leq\frac{\sqrt{3}}{\pi}$
Fox-Goodman	1/2	1/12	$\frac{\Delta t}{T}\leq\frac{\sqrt{3/2}}{\pi}$
Parabola	2/3	49/144	Unconditionally Stable

On the Selection of a Numerical Integration Method

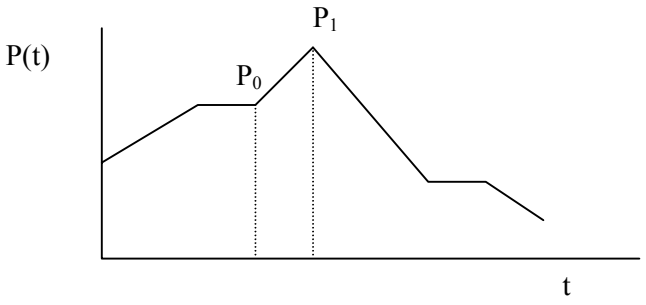
If stability is an issue (say we have a large F.E.M. model with many modes), then we want an unconditionally stable method. Another issue is how the method behaves regarding period distortion and numerical damping.



Typically, numerical damping depends on $\frac{\Delta t}{T}$.

Exact numerical integration method for SDOF systems with piece-wise linear excitation

The basic idea in this approach is to obtain the exact solution for a SDOF system for a linearly varying load and to use the solution to set up a marching algorithm. While one can in principle have time stations only at the locations where the load changes slope, it is customary to advance using equally spaced time steps.



At $t = t_1$ $\qquad m\ddot{u}_1 + c\dot{u}_1 + ku_1 = P_1$

At $t = t_o\, t_0$ $\qquad m\ddot{u}_0 + c\dot{u}_0 + ku_0 = P_0$

Taking the difference one gets

$$m\Delta u(\tau) + c\Delta u(\tau) + k\Delta u(\tau) = \frac{\Delta P\tau}{\Delta t} \tag{1}$$

We find the exact solution to Equation (1) and evaluate it at $\tau = \Delta t$. The result can be written: in the form;

$$\Delta u = AP_0 + BP_1 + (C-1)u_0 + D\dot{U}_0$$

$$\Delta\dot{u} = A'P_0 + B'P_1 + C'U_0 + (D'-1)\dot{U}_0$$

$$\begin{Bmatrix} u \\ \dot{u} \end{Bmatrix} = \begin{bmatrix} A & B \\ A' & B' \end{bmatrix} \begin{Bmatrix} P_0 \\ P_1 \end{Bmatrix} + \begin{bmatrix} C-1 & D \\ C' & D'-1 \end{bmatrix} \begin{Bmatrix} u_0 \\ \dot{u}_0 \end{Bmatrix}$$

where the constants are given in Appendix A.

STATE FORM OF THE EQUATIONS OF MOTION

This section concludes the part of the course where we look at techniques for solving the equations of motion. The essential idea of the state form is that of reducing the system of equations from second order to first order. As will be shown, this can be done at the expense of increasing the DOF in the solution from N to 2N. The first order equations offer many advantages, including the possibility for obtaining a closed form (albeit implicit) expression for the response of a MDOF system to arbitrary loading.

Consider the equations of motion of a linear viscously damped MDOF system;

$$[m]\{\ddot{u}\}+[c]\{\dot{u}\}+[k]\{u\}=\{P(t)\}$$

define

$$\{\dot{u}\}=\{x\}$$

and get

$$[m]\{\dot{x}\}+[c]\{x\}+[k]\{u\}=\{P(t)\}$$

where

$$\{\dot{x}\}=[m]^{-1}(\{P(t)\}-[c]\{x\}-[k]\{u\})$$

the above equations can be conveniently written as;

$$\begin{Bmatrix}\dot{u}\\ x\end{Bmatrix}=\begin{bmatrix}0 & I\\ -m^{-1}k & -m^{-1}c\end{bmatrix}\begin{Bmatrix}u\\ x\end{Bmatrix}+\begin{Bmatrix}0\\ [m]^{-1}\{P(t)\}\end{Bmatrix}$$

the vector that contains the displacements and velocities is the state vector, which we designate as $\{y\}$. We can write; $\{y\}=\begin{Bmatrix}u\\ x\end{Bmatrix}$

$$\{\dot{y}\}=[A]\{y\}+[B]\{f(t)\}$$

where [A] is the system's matrix given by

$$[A]=\begin{bmatrix}0 & I\\ -m^{-1}k & -m^{-1}c\end{bmatrix}$$

and [B] and $\{f(t)\}$ are given by;

$$[B] = \begin{bmatrix} 0 & 0 \\ 0 & [m]^{-1} \end{bmatrix}$$

$$\{f(t)\} = \begin{Bmatrix} 0 \\ P(t) \end{Bmatrix}$$

Now consider the solution of

$$\dot{y} = Ay + Bf(t) \tag{1}$$

Consider multiplying eq.1 by an arbitrary matrix K(t), we get

$$K(t)\,\dot{y} = K(t)Ay + K(t)Bf(t) \tag{2}$$

and since

$$\frac{d}{dt}\big(K(t)y\big) = K(t)\,\dot{y} + \dot{K}(t)y \tag{3}$$

it follows from (3) that

$$K(t)\,\dot{y} = \frac{d}{dt}(K(t)y) - \dot{K}(t)y \tag{4}$$

substituting (4) into (2)

$$\frac{d}{dt}(K(t)y) - \dot{K}(t)y = K(t)Ay + K(t)Bf(t) \tag{5}$$

we now impose the condition;

$$K(t)A = -\dot{K}(t) \tag{6}$$

and get

$$\frac{d}{dt}(K(t)y) = K(t)Bf(t) \tag{7}$$

Integrating both sides of eq.7 yields

$$K(t)y = \int_0^t K(\tau)Bf(\tau)d\tau + C \tag{8}$$

at t = 0 we have;

$$K_0 y_0 = 0 + C \Rightarrow C = K_0 y_0$$

and we can write

$$K(t)y(t) = \int_0^t K(\tau)Bf(\tau)d\tau + K_0 y_0 \tag{9}$$

multiplying eq.9 by $K(t)^{-1}$ gives

$$y(t) = \int_0^t K(t)^{-1} K(\tau)Bf(\tau)d\tau + K(t)^{-1} K_0 y_0 \tag{10}$$

Lets examine the implications of eq.6. Assuming that the matrices K(t) and A commute we can write;

$$AK(t) = -\dot{K}(t) \tag{11}$$

eq.11 is of the form

$$\dot{u} = -\alpha u$$

which has the solution

$$u = be^{-\alpha t}$$

therefore, the solution to eq.11 is;

$$K(t) = be^{-At} \tag{12}$$

at t=0

$$K(0) = b$$

so the solution is

$$K(t) = K_0 e^{-At} \tag{13}$$

substituting eq.13 into eq.10 we conclude that

$$y = \int_0^t K_0^{-1} e^{At} e^{-A\tau} K_0 B f(\tau) d\tau + K_0^{-1} e^{At} K_0 y_0 \qquad (14)$$

since K_0 is arbitrary we take it equal to the identity and write;

$$y = \int_0^t e^{A(t-\tau)} B f(\tau) d\tau + e^{At} y_0 \qquad (15)$$

Proof of assumption about A and K(t) being matrices that commute.

$$K(t) = e^{-At}$$

From a Taylor Series expansion about the origin it follows that

$$K(t) = e^{-At} = I - At + \frac{t^2 A^2}{2!} - \frac{t^3 A^3}{3!} + \ldots \qquad (16)$$

from where it is evident that pre and post multiplication by A lead to the same result. W*e are justified, therefore, in switching K(t)A to AK(t).*

Also as one gathers from the Taylor Series expansion, e^{At} is accurate when t is not large since as t increases we need more and more terms of the series. Nevertheless, we can always use the following technique

$$e^{At} = e^{A(\Delta t1 + \Delta t2 + \ldots.. \Delta tn)} = e^{A\Delta 1} e^{A\Delta t2} \ldots.. e^{A\Delta tn}$$

Where all the terms are accurate.

In addition it is customary to call $e^{At} = \Phi(t)$ = **Trans Matrix.**
Therefore substituting, reversing, and folding (15), it follows that

$$y = \int_0^t \Phi(\tau) B f(t-\tau) d\tau + \Phi(t) y_0 \quad (17)$$

Evaluation of $\Phi(t)$.

$$\Phi(t) = e^{At} = \sum_{n=o}^{\infty} \frac{t^n A^n}{n!} \text{ this series always converges.}$$

Note that

$$\Phi(t) = e^{At} = e^{An\Delta t} = (e^{A\Delta t})^n \text{ which can be used to improve accuracy.}$$

If the matrix $e^{A\Delta t}$ has a spectral radius greater than 1, then as n increases $\Phi(t)$ increases and the initial conditions grow indicating the system is unstable.

In summary, Stability of a physical linear system requires that the spectral radius of $\Phi(t) < 1$ (for any t).

Numerical Implementation.

It follows from (17) that

$$y(t+\Delta t) = \int_0^{\Delta t} \Phi(\tau) B f(t-\tau) d\tau + \Phi_{(\Delta t)} y(t)$$

Let's use the simple approach, which is explained by the following equation

$$y(t+\Delta t) = \Phi_{(\frac{\Delta t}{2})} B f_{(t+\frac{\Delta t}{2})} + \Phi_{(\Delta t)} y(t) \quad (18)$$

The advantage of (18) over more complicated forms is the fact that no inversion is required.

We only need to compute the matrices

$$\Phi_{(\frac{\Delta t}{2})} = e^{A\Delta t/2} \text{ and } \Phi_{(\Delta t)} = e^{At}$$

A more refined way.

Assume the load within the interval is approximated as a constant evaluated at some interior point say

$$t + n\Delta t \text{ where } (0 < n < 1)$$

It follows that

$$y(t+\Delta t) = \int_0^{\Delta t} e^{A\tau} (B f(t+n\Delta t) d\tau + \Phi_{(\Delta t)} y(t) \quad (19)$$

And evaluating the integral we get

$$y(t+\Delta t) = A^{-1} e^{A\tau} l_0^{\Delta t} B f(t+n\Delta t) + \Phi_{(\Delta t)} y(t)$$

And evaluating the limits it follows that

$$y(t+\Delta t) = A^{-1}[e^{A\tau} - I]Bf(t+n\Delta t) + \Phi_{(\Delta t)}y(t) \quad (20)$$

Exact Solution for Piece-Wise Linear Loading by the Transition Matrix Approach

$$Y(t)=\int_0^t e^{A(t-\tau)}Bf(\tau)d\tau + e^{At}Y_0 \quad (1)$$

assume

$$f(\tau)= f_0 + \dot{f}\tau \quad (2)$$

The derivation to be presented involves the integral,

$$\int_0^t e^{at}\tau d\tau \quad (3)$$

we solve the integral first to allow a smoother flow in the derivation later;

Integrating by parts :

$$e^{at}dt= dv \Rightarrow a^{-1}e^{at}$$

$$t= u \Rightarrow dv= dt$$

$$\int_0^t e^{at}\tau d\tau = a^{-1}e^{at}\tau \Big|_0^t - \int_0^t a^{-1}e^{at}d\tau$$

$$\int_0^t e^{at}\tau d\tau = a^{-1}e^{at}\tau - (a^{-1})^2 e^{at} + (a^{-1})^2 \quad (4)$$

Consider now eq.1 with the linear load variation described by eq.2,

$$Y(t)=\int_0^t e^{At}e^{-A\tau}B(f_0 + \dot{f}\tau)d\tau + e^{At}Y_0 \quad (5)$$

one can clearly write

$$Y(t)=\int_0^t e^{At}e^{-A\tau}Bf_0 d\tau + \int_0^t e^{At}e^{-A\tau}B\dot{f}\tau\, d\tau + e^{At}Y_0 \quad (6)$$

solving the first integral one gets;

$$Y(t)=e^{At}(-A)^{-1}\left[e^{-A\tau}\right]_0^t Bf_0 + \int_0^t e^{At}e^{-A\tau}B\dot{f}\tau d\tau + e^{At}y_0 \qquad (7)$$

or

$$Y(t)=e^{At}(-A)^{-1}\left[e^{-At}-I\right]Bf_0 + \int_0^t e^{At}e^{-A\tau}B\dot{f}\tau d\tau + e^{At}y_0 \qquad (8)$$

Since e^{At} and A^{-1} commute we can write;

$$Y(t)=(-A)^{-1}\left[I-e^{At}\right]Bf_0 + \int_0^t e^{At}e^{-A\tau}B\dot{f}\tau d\tau + e^{At}y_0$$

$$Y(t)=\underbrace{A^{-1}\left[e^{At}-I\right]Bf_0}_{P_1} + \underbrace{\int_0^t e^{At}e^{-A\tau}B\dot{f}\tau\, d\tau}_{e^{At}B \text{ and } \dot{f} \text{ are constant}} + e^{At}y_0$$

After taking the constants out of the integral the form is that in eq.4. In this case a = -A and t = τ , we find;

$$Y(t)=P_1 f_0 + e^{At}\left[t(-A)^{-1}e^{-At} - \left(A^{-1}\right)^2 e^{-At} + \left(A^{-1}\right)^2\right]B\dot{f} + e^{At}y_0 \qquad (9)$$

$$Y(t)=P_1 f_0 + \left[t(-A)^{-1} - \left(A^{-1}\right)^2 + \left(A^{-1}\right)^2 e^{At}\right]B\dot{f} + e^{At}y_0$$

$$Y(t)=P_1 f_0 + A^{-1}\left(A^{-1}\left(e^{At}-I\right)\right)B\dot{f} - tA^{-1}B\dot{f} + e^{At}y_0$$

which can be written as

$$Y(t)=P_1 f_0 + A^{-1}P_1\dot{f} - tA^{-1}B\dot{f} + e^{At}y_0 \qquad (10)$$

Eq.10 can be further simplified by recognizing that

$$\dot{f}=\frac{f_1-f_0}{t} \qquad (11a)$$

and

$$f_1 = f_{(t)} \qquad (11b)$$

substituting eq.11 into eq.10 one gets

$$Y(t) = P_1 f_0 + \frac{A^{-1}P_1(f_1 - f_0)}{t} - \frac{tA^{-1}B(f_1 - f_0)}{t} + e^{At}y_0$$

$$Y(t) = P_1 f_0 + \frac{A^{-1}P_1 f_1}{t} - \frac{A^{-1}P_1 f_0}{t} - A^{-1}Bf_1 + A^{-1}Bf_0 + e^{At}y_0$$

$$Y(t) = \left(P_1 - \frac{A^{-1}P_1}{t} + A^{-1}B\right)f_0 + \left(\frac{A^{-1}P_1}{t} - A^{-1}b\right)f_1 + e^{At}y_0$$

$$Y(t) = \underbrace{\left(P_1 + A^{-1}\left(B - \frac{P_1}{T}\right)\right)}_{P_2} f_0 - \underbrace{A^{-1}\left(B - \frac{P_1}{t}\right)}_{P_2} f_1 + e^{At}y_0$$

So the result can be written as;

$$Y(t) = (P_1 + P_2)f_0 - P_2 f_1 + e^{At}y_0 \qquad (12)$$

MATLAB IMPLEMENTATION

A MATLAB routine that implements the transition matrix approach is presented in this section.

```
%*************************************************************
%
%                          ***** TMA *****
%
%                Prepared by D.Bernal - February, 1999.
%
%
%*************************************************************
% Program computes the response of linear MDOF systems using the
% transition matrix approach. The basic assumption is that the
% load varies linearly within a time step. Except for the
% approximation in the computation of the exponential matrix, the
% solution is exact for the assumed load variation.

% INPUT
```

```
% m,c,k = mass dampig and stiffness matrices for the system.
% dt = time step
% g = vector that scales the applied load at each DOF
% p = load matrix. Each row contains the time history of the load
% applied at each DOF. Since the case where the loads have the
% same history at each DOF is common, an option to input p as a
% single row is provided. In particular, if p has a single row,
% the program assumes that this row applies to all the DOF.
% d0,v0 = initial conditions
% nsteps = number of time steps in the solution.

% OUTPUT
% u = matrix of displacements
% ud = matrix of velocities
% udd = matrix of accelerations (if flag=7);
% (each column contains one DOF);

% LIMITATIONS
% Mass matrix must be positive definite
%*************************************************************

function [u,ud,udd]= tma(m,c,k,dt,g,p,d0,v0,nsteps);

% To eliminate the computation of accelerations flag~=7;
flag=7;

% Basic matrices.
[dof,dum]=size(m);
A=[eye(dof)*0 eye(dof);-inv(m)*k -inv(m)*c];
B=[eye(dof)*0 eye(dof)*0;eye(dof)*0 inv(m)];

% Place d0,v0 and g in column form in case they where input as a
% row vectors and make sure that the load is ordered in rows;
[r,c]=size(g);
if r==1;
g=g';
end;
[r,c]=size(d0);
if r==1;
d0=d0';
end;
[r,c]=size(v0);
if r==1;
v0=v0';
end;
[r,c]=size(p);
if r~=dof;
if r~=1;
p=p';
end;
end;
```

```
% Compute the matrices used in the marching algorithm.
AA=A*dt;
F=expm(AA);
X=inv(A);
P1=X*(F-eye(2*dof))*B;
P2=X*(B-P1/dt);


% Initialize.
yo=[d0;v0];
%Perform the integration.
for i=1:nsteps;
y1(:,i)=(P1+P2)*[g*0;p(:,i).*g]-
P2*[g*0;p(:,(i+1)).*g]+F*yo;yo=y1(:,i);
end;
u=y1(1:dof,:)';
u=[d0';u];ud=y1(dof+1:2*dof,:)';
ud=[v0';ud];
% Calculate the acceleration vector if requested.
if flag==7;
X=inv(m);
for i=1:nsteps+1;
uddi=X*(p(:,1).*g-c*ud(i,:)'-k*u(i,:)');
udd=[udd uddi];
end;
udd=udd';
end;
```

Connection Between e^{At} and the Impulse Response.

In the initial treatment of the impulse response we limited our attention to a SDOF system. By examining the general solution for the transition matrix it is evident that the matrix e^{At} plays the role of the impulse response in the case of MDOF systems. Two results that may be appropriate to note explicitly are:

1) If a load function has a constant spatial distribution the response at DOF "j" can be obtained by the convolution of a single "impulse response function" with the time history of the load. The impulse response functions for each one of the DOF are given by:

$$\{h(t)\} = e^{At}\begin{Bmatrix} 0 \\ m^{-1}g \end{Bmatrix} \tag{13}$$

where *m* is the mass matrix and *g* is vector that describes the spatial distribution of the loading. In this case one can think about the response at DOF i as that of a SDOF system with frequency dependent parameters subjected to the load history. The frequency dependent parameters are obtained from the Fourier Transform of the appropriate impulse response function.

2) In the more general case the load can be expressed as

$$P(t) = \sum_{i=1}^{k} g_i \; p_i(t) \tag{14}$$

where k ≤ #DOF. As is evident, the response at each DOF is now given by the superposition of "k" convolutions with impulse response functions defined by eq.13. As a generalization of the statement that concluded the previous entry, we can now visualize the response at DOF "j" as the sum of the response of "k" frequency dependent SDOF systems with parameters defined by the Fourier transforms of the corresponding impulse response functions.

3) Equivalent to (2) is the idea that one can define an impulse response matrix where the $h_{i,j}$ is the response at DOF i due to an impulse at DOF j. By inspection of eqs. 13 and 14 one concludes that $h_{i,j}$ is the "ith" term in eq.13 when the load distribution g_j is unity at DOF "j" and zero elsewhere.

4) It is also of interest to note that the impulse response matrix is the upper dof x dof partition of the matrix e^{At} (remember this matrix is 2dof x 2dof). Inspecting the Taylor series expansion of e^{At} one can write a series expression for the impulse response matrix – the first few terms are:

$$[h] = [I]\, t - [m]^{-1}[k]\frac{t^2}{2} + (-[m]^{-1}[c] + ([m]^{-1}[k])^2)\frac{t^3}{6} + \cdots \tag{15}$$

which shows that as $t \to 0$ the impulse response approaches the identity times t – note also that the off-diagonal terms $\to 0$.

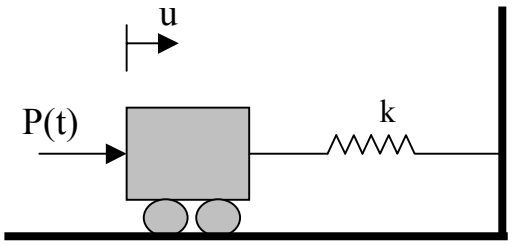
PART II

In the first part of the course the focus was in the solution of the equations of motion but little was said about how these equations are obtained. In this section of the course we focus on the formulation of the equations for elastic MDOF systems.

Introduction to Analytical Mechanics

One approach to get the equations of motion for a systems is by using Newton's law.

i.e.



From a free body diagram of the mass one gets;

$$m\ddot{u} + ku = P(t$$

While one can always obtain the equations of motion using equilibrium considerations, the direct application of Newton's laws is difficult in complex systems due to the fact that the quantities involved are vectors. An alternative approach is to use energy concepts – the energy approach is treated in the branch of mechanics known as Analytical Mechanics.

Important Contributors to Analytical Mechanics:

Leibnitz : Noted that equation of motion could also be expressed in terms of energy.

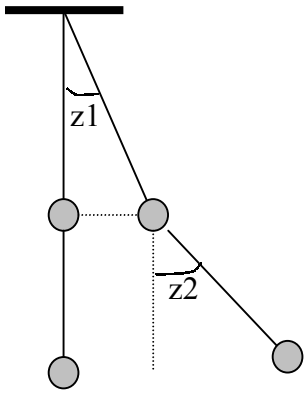
D'alembert : First to apply the principle of virtual work to the dynamic problem.

Euler: Derived (among many other things) the calculus of variations.

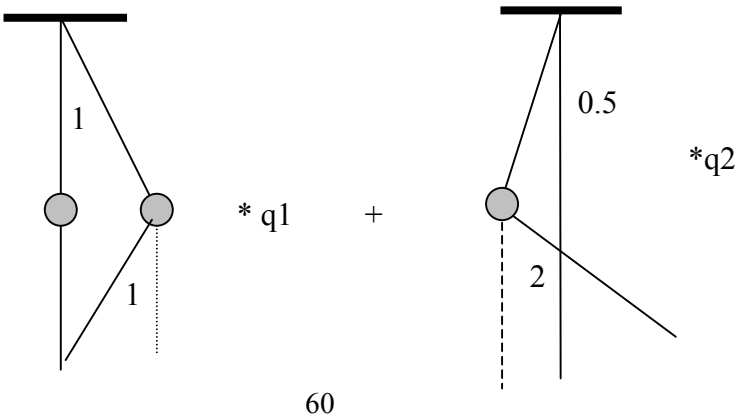
Hamilton: Hamilton's Principle is the cornerstone of Analytical Dynamics.

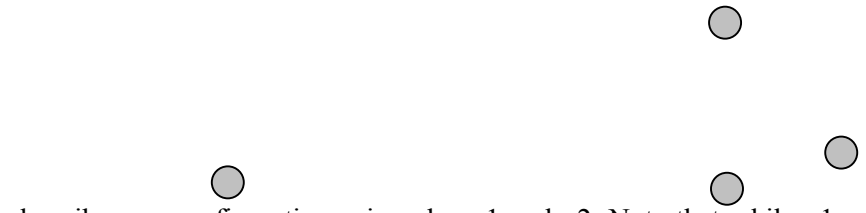
Langrange: His application of Hamilton's principle to the case where the displacement field can be expressed in terms of generalized coordinates leads to a convenient form of the principle known as Langrange's Equation.

Generalized Coordinates (Langrange)



Alternatively

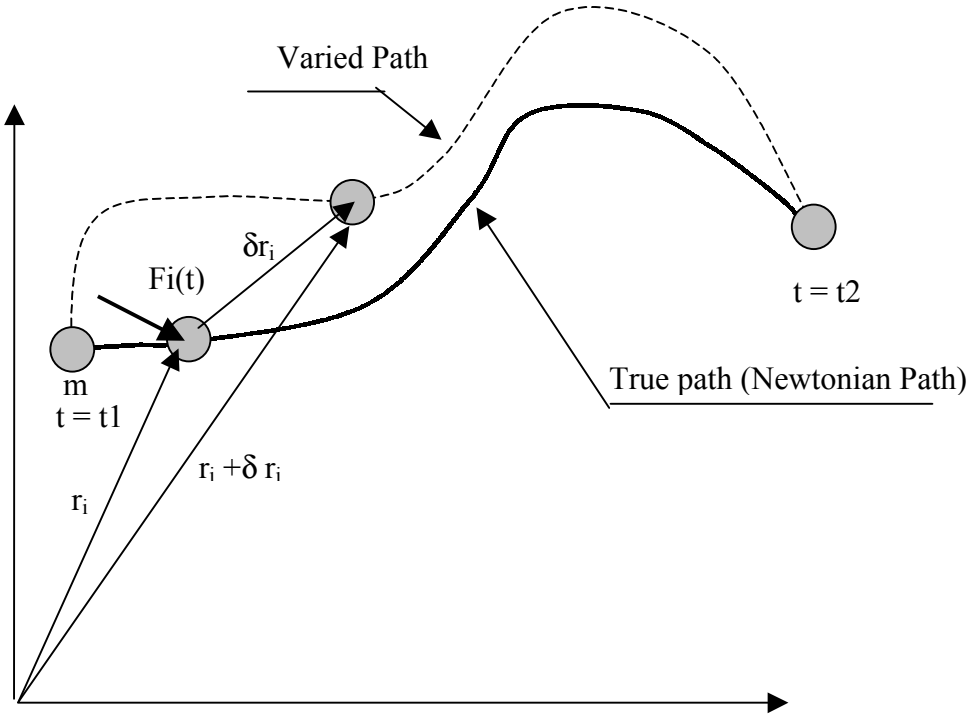




We can describe any configuration using also q1 and q2. Note that while q1 and q2 can not be measured, they are coordinates that describe the configuration and are just as valid as the physical coordinates. Generalized coordinates are any set of parameters that fully described the configuration of the system.

HAMILTON'S PRINCIPLE

Consider a particle that has moved from location 1 to location 2 under the action of a resultant force $F_i(t)$. The true path and alternative (incorrect path) are noted in the figure.



From Newton's Law we can write:

$$F_i(t) = m_i \ddot{r}_i \tag{1}$$

or, placing it in the form of an equilibrium equation we get;

$$F_i(t) - m_i \ddot{r}_i = 0 \tag{2}$$

Applying the principle of virtual work we can write (note that this is done with time frozen);

$$\left(F_i(t) - m_i \ddot{r}_i\right)\delta r_i = 0 \tag{3}$$

Consider the following identity;

$$\frac{d}{dt}\left[(m_i \dot{r}_i)\delta r_i\right] = m_i \ddot{r}_i \delta r_i + m_i \dot{r}_i \delta \dot{r}_i \tag{4}$$

from where it is evident that we can write

$$m_i \ddot{r}_i \delta r_i = \frac{d}{dt}\left[(m_i \dot{r}_i)\delta r_i\right] - m_i \dot{r}_i \delta \dot{r}_i \tag{5}$$

substituting eq.5 into eq.3 we get

$$F_i(t)\delta r_i - \frac{d}{dt}\left[(m_i \dot{r}_i)\right]\delta r_i + m_i \dot{r}_i \delta \dot{r}_i = 0 \tag{6}$$

Noting that;

$$\delta\left(\frac{1}{2} m_i \dot{r}_i \dot{r}_i\right) = \frac{1}{2} m_i (\dot{r}_i + \delta \dot{r}_i)(\dot{r}_i + \delta \dot{r}_i) - \frac{1}{2} m_i \dot{r}_i \dot{r}_i \tag{7}$$

or what is the same;

$$\ddot{a}\left(\frac{1}{2} m_i \dot{r}_i \dot{r}_i\right) = \frac{1}{2} m_i \dot{r}_i \dot{r}_i + \frac{1}{2} m_i \dot{r}_i \delta \dot{r}_i + \frac{1}{2} m_i \delta \dot{r}_i \dot{r}_i + \frac{1}{2} m_i \delta \dot{r}_i \delta \dot{r}_i - \frac{1}{2} m_i \dot{r}_i \dot{r}_i \tag{8}$$

allows one to write, neglecting higher order terms;

$$\delta\left(\frac{1}{2}(m_i \dot{r}_i)\dot{r}_i\right) = m_i \dot{r}_i \delta \dot{r}_i \tag{9}$$

substituting eq.9 into eq.6 gives

$$F_i(t)\delta r_i - \frac{d}{dt}\left[(m_i \dot{r}_i)\delta r_i\right] + \delta\left(\frac{1}{2} m_i \dot{r}_i \dot{r}_i\right) = 0 \tag{10}$$

We recognize the term $F_i(t)\delta r_i$ as the work done by F_i on the arbitrary virtual displacement at time t, we designate this term $\delta \overline{W}$. Furthermore, the third term is the variation of the kinetic energy which we designate as δT . In equation form, we have;

$$\delta \overline{W} = F_i(t)\delta r_i \tag{11}$$

and

$$\delta T = \delta\left(\frac{1}{2} m_i \dot{r}_i \dot{r}_i\right) \tag{12}$$

Substituting eqs.11 and 12 one can write eq.10 as;

$$\delta\overline{W} + \delta T = \frac{d}{dt}\left[m_i \dot{r}_i \delta r_i\right] \tag{13}$$

Multiplying both sides of eq.13 by dt and integrating from t1 to t2 one gets;

$$\int_{t_1}^{t_2}\left(\delta \overline{W} + \delta T\right)dt = m_i \dot{r}_i \delta r_i \Big|_{t_1}^{t_2} = 0 \tag{14}$$

where the equality to zero is justified because $\delta r_1 = \delta r_2 = 0$ at t1 and t2 (recall that a requirement imposed at the start of the derivation is that the varied path starts and ends on the actual path).

In summary;

$$\int_{t_1}^{t_2}\left(\delta \overline{W} + \delta T\right)dt = 0 \tag{15}$$

which is the mathematical statement of Hamilton's Principle. As can be seen, the principle states that the variation of the work of all the forces acting on the particle and the variations of the kinetic energy integrate to zero between any two points that are located on the true path.

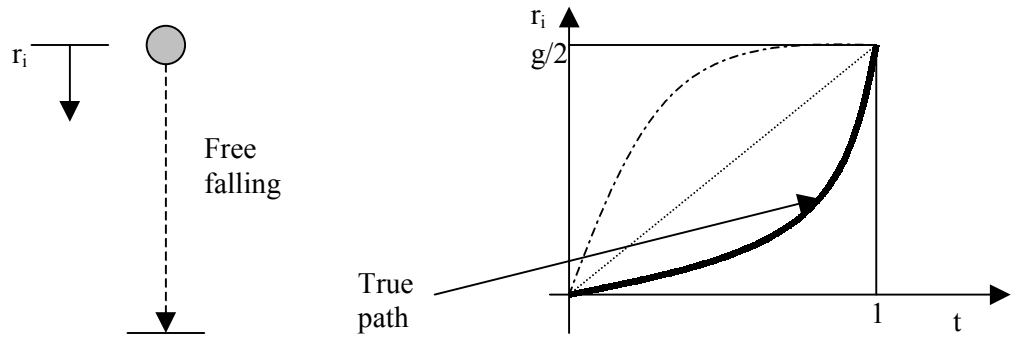
Provided that the varied paths considered are physically possible (do not violate constraints) the variation and the integration processes can be interchanged. For this condition eq.15 can be written as;

$$\delta\int_{t_1}^{t_2}\left(\overline{W} + T\right)dt = 0 \tag{16}$$

which is Hamilton's principle in its most familiar form. The principle states: *The actual path that a particle travels renders the value of the definite integral in eq.16 stationary with respect to all arbitrary variations between two instants t1 and t2, provided that the path variation is zero at these two times.*

Illustrative Example

Consider the case of a mass that falls freely.



Consider a family of paths that contains the correct solution.

$$\text{say} \quad r_i = \frac{g}{2} t^n$$

where we impose the limitation t1=0 and t2 = 1 to ensure that the varied paths and the true paths coincide at the limits of the time span; differentiating we get;

$$\dot{r}_i = n \frac{g}{2} t^{n-1}$$

the work and kinetic energy expressions are, therefore;

$$\overline{W} = m \frac{g^2}{2} t^n$$

and

$$T = \frac{1}{2} m \left(\frac{g}{2} n \right)^2 t^{2(n-1)}$$

substituting in eq.16

$$\delta \int_{t_1}^{t_2} \left(\overline{W} + T \right) dt = \delta \left[\int_0^1 m \frac{g^2}{2} t^n dt + \int_0^1 \frac{1}{2} m \frac{g^2}{4} n^2 t^{2(n-1)} dt \right] = 0$$

the definite integral equals

$$\int_{t_1}^{t_2}\left(\overline{W}+T\right)dt=\frac{mg^2}{2}\left[\frac{1}{n+1}+\frac{n^2}{4(2n-1)}\right]$$

In this simple case the requirement that the variation of the integral be zero is equivalent to requiring that the derivative with respect to *n* be zero. Taking a derivative and equating to zero one finds n = 2 which is, of course, the correct solution.

$$\text{Assume now } r_i=\frac{g}{2\left(1-e^n\right)}\left(1-e^{nt}\right)$$

which, of course, is a family that does not contain the correct path, we have;

$$\dot{r}_i=-\frac{gn}{2\left(1-e^n\right)}e^{nt}$$

therefore

$$\overline{W}=\frac{m_i g^2\left(1-e^{nt}\right)}{2\left(1-e^n\right)}$$

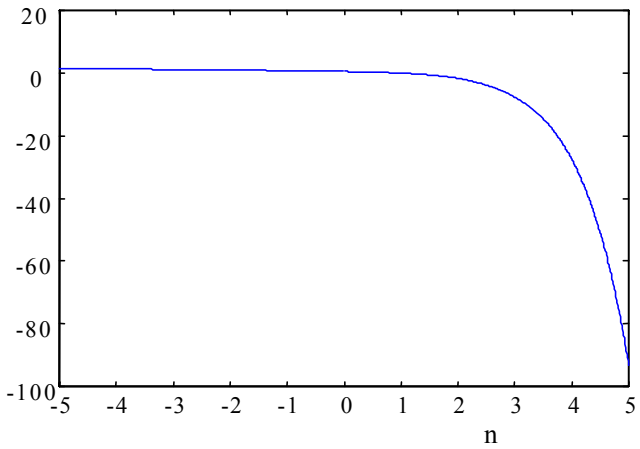
and

$$T=\frac{1}{2}\frac{m_i g^2 n^2 e^{2nt}}{4\left(1-e^n\right)^2}$$

the definite integral is therefore;

$$H=\frac{m_i g^2}{2\left(1-e^n\right)}\left[1+\frac{1-e^n}{n}+\frac{n\left(e^{2n}-1\right)}{8}\right]$$

A plot of this function from n = -100 to 100 is shown in the figure below, as expected, no stationary points are found.



Lagrange's Equations:

The derivation of Lagrange's equations starts by considering Hamilton's Principle, namely;

$$\int_{t_1}^{t_2} \left(\ddot{a}\overline{W} + \ddot{a}T \right) dt = 0 \tag{17}$$

For convenience assume that W is separated into Wc + Wnc where Wc is the work done by conservative forces, (gravity field for example) and Wnc is the work from non-conservative forces, which is typically that which comes from applied loads and damping. Defining Wc = -V, where V is called the potential, one can write;

$$\int_{t_1}^{t_2} \left(\ddot{a}Wnc - \ddot{a}V + \ddot{a}T \right) dt = 0 \tag{18}$$

The key step in the derivation of Lagrange's equations is the assumption that the terms inside the parenthesis in Hamilton's equation can be expressed as functions of generalized coordinates in the following way,

$$T = T(\dot{q}_1, \dot{q}_2, ..., \dot{q}_n, q_1, q_2, ..., q_n) \tag{19}$$

$$V = V(q_1, q_2, ..., q_n) \tag{20}$$

and

$$\ddot{a}\, Wnc = \sum Q_i \ \delta q_i \tag{21}$$

where we note that the kinetic energy is not just a function of velocities but may also have terms that depend on displacements. Taking the variation of T and V one gets;

$$\ddot{a}\, T = \sum_{i=1}^{N} \frac{\partial T}{\partial \dot{q}_i} \delta \dot{q}_i + \frac{\partial T}{\partial qi} \delta q_i \tag{22}$$

$$\ddot{a}\, V = \sum_{i=1}^{N} \frac{\partial V}{\partial qi} \delta q_i \tag{23}$$

substituting eqs.22 and 23 into 17 gives;

$$\int_{t_1}^{t_2} \left(\sum_{i=1}^{N} \frac{\partial T}{\partial \dot{q}_i} \delta \dot{q}_i + \frac{\partial T}{\partial qi} \delta q_i - \frac{\partial V}{\partial q_i} \delta qi + Q_i \delta q_i \right) dt = 0 \tag{24}$$

we wish to eliminate the derivative with respect to the variation of the velocity to allow the variation with respect to displacement to be taken common factor of the terms in the parenthesis; we can do this with an integration by parts of the integral

$$\int_{t_1}^{t_2} \frac{\partial T}{\partial \dot{q}_i} \delta \dot{q}_i dt \tag{25}$$

we take

$$U = \frac{\partial T}{\partial \dot{q}_i}$$

and

$$dV = \delta \dot{q}_i dt$$

therefore

$$dU = \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) dt$$

and

$$V = \delta q_i$$

and we get

$$\left[\frac{\partial T}{\partial \dot{q}_i} \delta q_i \right]_{t1}^{t2} - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) \delta q_i dt \tag{26}$$

where the first term is zero because the variations vanish at t = t1 and t = t2. Substituting eq.26

into eq.24 one gets;

$$\int_{t_1}^{t_2}\left(\left(\sum_{i=1}^{N}-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)+\frac{\partial T}{\partial q_i}-\frac{\partial V}{\partial q_i}+Q_i\right)\delta q_i\right)dt=0 \qquad (27)$$

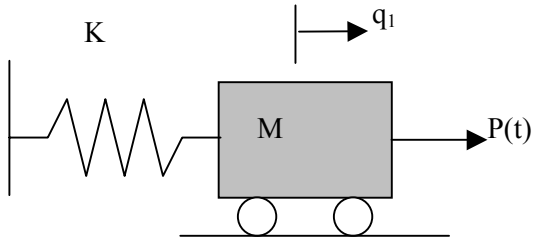
Since the variations δqi are independent we can choose all of them to equal zero except for one. In this way we reach the conclusion that the only way the integral can be zero is if each of the individual terms in the bracket vanish. In particular, we must require that;

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial V}{\partial q_i}=Q_i \qquad (28)$$

for every i. Eqs.28 are the much celebrated Lagrange's Equations.

Example

Consider the SDOF system shown in the figure;



the necessary expressions for obtaining the equation of motion using eq.28 are:

$$T=\frac{1}{2}m\dot{q}_1^{\,2}$$

$$Wc=-\int_0^{q_1}Kq_1dq_1=-\frac{Kq_1^{\,2}}{2}$$

$$V=\frac{1}{2}Kq_1^{\,2}$$

$$\delta Wnc=P(t)\delta q_1$$

The mathematical operations are straightforward and are presented next without comment.

$$\frac{\partial T}{\partial \dot{q}_1} = m\dot{q}_1$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = m\ddot{q}_1$$

$$\frac{\partial T}{\partial q_1} = 0$$

$$\frac{\partial V}{\partial q_1} = Kq_1$$

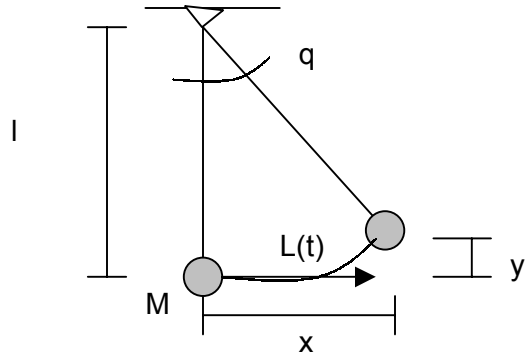
Substituting in eq.28 one gets;

$$m\ddot{q}_1 + Kq_1 = P(t)$$

which is the equation of motion for the system.

Example

Consider the pendulum shown in the figure.



$$x = \ell \sin q$$

$$y = \ell - \ell \cos q$$

$$\dot{x} = \ell \dot{q} \cos q$$

$$\dot{y} = \ell \dot{q} \sin q$$

The expressions for the kinetic energy, the potential, and the work of the non-conservative forces are;

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 = \frac{1}{2}m(\ell^2\dot{q}^2\cos^2 q + \ell^2\dot{q}^2\sin^2 q) = \frac{1}{2}m\ell^2\dot{q}^2$$

$$V = mg\ell(1 - \cos q)$$

$$\delta Wnc = L(t)\delta x = L(t)\ell(\sin(q+\delta q) - \sin q) = L(t)\ell(\sin q\cos q + \cos q\sin\delta q - \sin q)$$
$$\delta Wnc = L(t)\ell\cos q\delta q$$

therefore,

$$\frac{\partial T}{\partial \dot{q}} = m\ell^2\dot{q}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) = m\ell^2\ddot{q}$$

$$\frac{\partial T}{\partial q} = 0$$

$$\frac{\partial V}{\partial q} = mg\ell\sin q$$

$$Q = L(t)\ell\cos q$$

$$m\ell^2\ddot{q} + mg\ell\sin q = L(t)\ell\cos q$$

or, deviding by the length

$$m\ell\ \ddot{q} + mg\sin q = L(t)\cos q$$

Note that the above equation is nonlinear. A linearized equation, however, can be obtained by restricting the validity of the results to small angles. In this case we can substitute the sine of the angle by the angle itself and the cosine by one, we get;

$$m\ell\ddot{q} + mg\,q = L(t)$$

which is a simple second order linear differential equation. The period of the pendulum for small amplitudes of vibration is readily obtained as;

$$\varpi = \sqrt{\frac{g}{\ell}}$$

$$T = \frac{2\pi}{\sqrt{\frac{g}{\ell}}}$$

Solution of Linear and Nonlinear Differential Equations by Conversion to First Order

As the example of the pendulum illustrates, the equations of motion are generally nonlinear and can only be linearized by restricting the solution to small displacements around the equilibrium configuration. Analytical solutions to nonlinear differential equations are not generally possible and one must almost always invoke a numerical technique to obtain the response for a particular set of conditions. In order to make future discussions more interesting it is useful to acquire the capability for solving, albeit numerically, the nonlinear equations that we obtain.

A powerful solution strategy consists in casting the equations in first order form. The procedure is readily understood by inspecting the example presented next.

Consider the case of the simple pendulum, the equation of motion was found to be;

$$m\ell\ \ddot{q} + mg\sin q = L(t)\cos q$$

we take

$x1 = q$

$x2 = \dot{q}$

and write the equation as;

$$ml\dot{x}_2 + mg\sin(x_1) = L(t)\cos(x_1)$$

from this equation and the fact that $\dot{x}_1 = x_2$ we can write the derivative of the state as;

$$\dot{x}_1 = x_2$$

and

$$\dot{x}_2 = \frac{1}{ml}(L(t)\cos(x_1) - mg\sin(x_1))$$

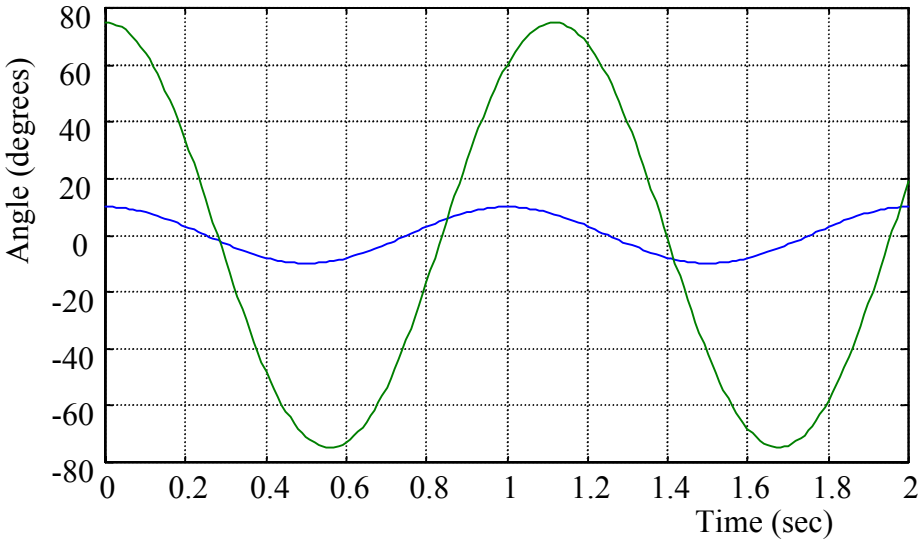
Clearly, knowing the gradient of the state one can estimate the increment in the response and proceed in a step by step fashion. In MATLAB the solution can be obtained using the numerical integrator ODE45 (there are others in MATLAB that may perform better for certain problems). The form of the command is:

[T,Y] = ODE45 ('name', tt, Y0);

where 'name' is the name of an m file provided by the user. This file must return a column vector with the derivative of the state when called with time and the state. tt = vector of times where the solution is desired and Y0 = initial state. The file pend.m presented next was prepared to integrate the equation for the pendulum. Note that the load L(t) would have to be specified inside of pend.m. In the file shown it has been taken equal to zero.

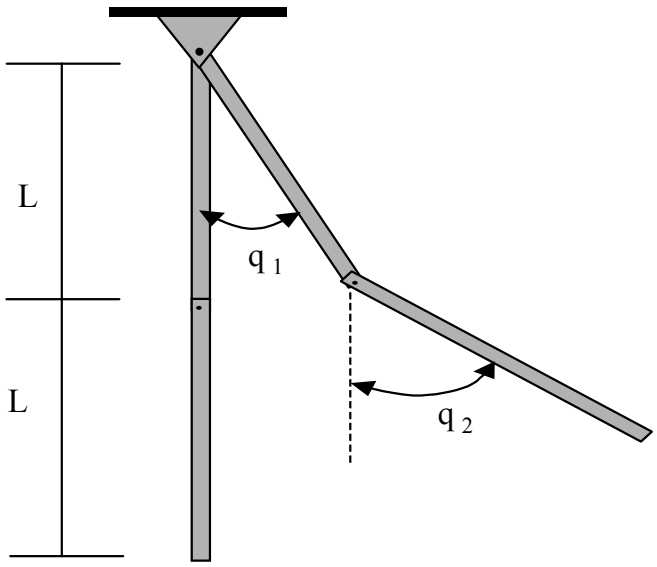
```
function [xd]=pend(T,X);
L=zeros(300,1);
g=386.4;
dt=0.01;
m=0.1;
l=g/4/pi^2;
n=round(T/dt+1);
X1D=X(2);
X2D=(L(n)*cos(X(1))-m*g*sin(X(1)))/(m*l);
xd=[X1D;X2D];
```

The free vibration response is plotted in the figure for an amplitudes of 10 and 75 degrees. As can be seen, the period for the low amplitude can not be distinguished from the theoretical answer obtained from the equation for small amplitudes (1 second). For the large release angle, however, the period is larger. It may be worth noting that while it is difficult to tell by inspection, the motion for the large amplitude is periodic but not harmonic.



Further Practice with Lagrange's Equations – The double Pendulum

The double pendulum shown in the figure has two DOF. In this section we derive the equations of motion and illustrate some of the interesting features of the behavior.



Equations of Motion

The equations of displacements for the double pendulum are :

$$x_1 = \frac{L}{2}\sin q_1$$

$$y_1 = \frac{L}{2}\cos q_1$$

$$x_2 = L\sin q_1 + \frac{L}{2}\sin q_2$$

$$y_2 = L\cos q_1 + \frac{L}{2}\cos q_2$$

If we take the first derivative of each of the above equations then we get :
So the kinetic energy with respect to x-axis, y-axis, and the rotational kinetic energy are :

$$\dot{x}_1 = \frac{L}{2}\dot{q}_1\cos q_1$$

$$\dot{y}_1 = \frac{-L}{2}\dot{q}_1\sin q_1$$

$$\dot{x}_2 = L\dot{q}_1\cos q_1 + \frac{L}{2}\dot{q}_2\cos q_2$$

$$\dot{y}_2 = -L\dot{q}_1\sin q_1 - \frac{L}{2}\dot{q}_2\sin q_2$$

$$T_{x,1} = \frac{1}{2} m \dot{x}_1^{\,2} = \frac{1}{2} m (\frac{L}{2} \dot{q}_1 \cos q_1)^2 = \frac{mL^2}{8} (\cos^2 q_1) \dot{q}_1^{\,2}$$

$$T_{x,2} = \frac{1}{2} m \dot{x}_2^{\,2} = \frac{1}{2} m (L\dot{q}_1 \cos q_1 + \frac{L}{2} \dot{q}_2 \cos q_2)^2 = \frac{mL^2}{2} (\dot{q}_1^{\,2} \cos^2 q_1 + \frac{\dot{q}_2^{\,2} \cos^2 q_2}{4} + \dot{q}_1 \dot{q}_2 \cos q_1 \cos q_2)$$

$$T_{y,1} = \frac{1}{2} m \dot{y}_1^{\,2} = \frac{1}{2} m (-\frac{L}{2} \dot{q}_1 \sin q_1)^2 = \frac{mL^2}{8} (\sin^2 q_1) \dot{q}_1^{\,2}$$

$$T_{y,2} = \frac{1}{2} m \dot{y}_2^{\,2} = \frac{1}{2} m (L\dot{q}_1 \sin q_1 + \frac{L}{2} \dot{q}_2 \sin q_2)^2 = \frac{mL^2}{2} (\dot{q}_1^{\,2} \sin^2 q_1 + \frac{\dot{q}_2^{\,2} \sin^2 q_2}{4} + \dot{q}_1 \dot{q}_2 \sin q_1 \sin q_2)$$

$$T_{rot,1} = \frac{1}{2} J \dot{q}_1^{\,2} = \frac{1}{2} \frac{mL^2}{12} \dot{q}_1^{\,2} = \frac{1}{24} mL^2 \dot{q}_1^{\,2}$$

$$T_{rot,2} = \frac{1}{2} J (\dot{q}_1 + \dot{q}_2)^2 = \frac{1}{2} \frac{mL^2}{12} (\dot{q}_1^{\,2} + \dot{q}_2^{\,2} + 2\dot{q}_1 \dot{q}_2) = \frac{1}{24} mL^2 (\dot{q}_1^{\,2} + \dot{q}_2^{\,2} + 2\dot{q}_1 \dot{q}_2)$$

The total kinetic energy will be :

$$T_{total} = T_{x,1} + T_{x,2} + T_{y,1} + T_{y,2} + T_{rot,1} + T_{rot2}$$

Doing the substitutions we get :

$$T_{total} = \frac{mL^2}{8} (\cos^2 q_1) \dot{q}_1^{\,2} + \frac{mL^2}{2} (\dot{q}_1^{\,2} \cos^2 q_1 + \frac{\dot{q}_2^{\,2} \cos^2 q_2}{4} + \dot{q}_1 \dot{q}_2 \cos q_1 \cos q_2) + \frac{mL^2}{8} (\sin^2 q_1) \dot{q}_1^{\,2}$$

$$+ \frac{mL^2}{2} (\dot{q}_1^{\,2} \sin^2 q_1 + \frac{\dot{q}_2^{\,2} \sin^2 q_2}{4} + \dot{q}_1 \dot{q}_2 \sin q_1 \sin q_2) + \frac{1}{24} mL^2 \dot{q}_1^{\,2} + \frac{1}{24} mL^2 (\dot{q}_1^{\,2} + \dot{q}_2^{\,2} + 2\dot{q}_1 \dot{q}_2) =$$

$$= \frac{mL^2}{8} (\sin^2 q_1 + \cos^2 q_1) \dot{q}_1^{\,2} + \frac{mL^2}{2} \dot{q}_1^{\,2} (\sin^2 q_1 + \cos^2 q_1) + \frac{mL^2}{8} \dot{q}_2^{\,2} (\sin^2 q_1 + \cos^2 q_1) +$$

$$+ \frac{mL^2}{2} \dot{q}_1 \dot{q}_2 (\sin q_1 \sin q_2 + \cos q_1 \cos q_2) + \frac{2}{24} mL^2 \dot{q}_1^{\,2} + \frac{1}{24} mL^2 \dot{q}_2^{\,2} + \frac{1}{12} mL^2 \dot{q}_1 \dot{q}_2 =$$

$$= \frac{17}{24} mL^2 \dot{q}_1^{\,2} + \frac{1}{6} mL^2 \dot{q}_2^{\,2} + mL^2 \dot{q}_1 \dot{q}_2 \frac{\cos(q_1 - q_2)}{2} + \frac{1}{12} mL^2 \dot{q}_1 \dot{q}_2$$

According to Lagrange's equations :

$$\frac{\mathrm{d}}{\mathrm{dt}} (\frac{\partial \mathrm{T}}{\partial \dot{\mathrm{q}}_\mathrm{i}}) - \frac{\partial \mathrm{T}}{\partial \mathrm{q}_\mathrm{i}} + \frac{\partial \mathrm{V}}{\partial \mathrm{q}_\mathrm{i}} = 0$$

Which for the first mass yields :

$$\frac{\partial \mathrm{T}}{\partial \dot{\mathrm{q}}_\mathrm{i}} = \frac{17}{24} (2\dot{\mathrm{q}}_1) \mathrm{mL}^2 + \frac{\cos(\mathrm{q}_1 - \mathrm{q}_2)}{2} (\dot{\mathrm{q}}_1) \mathrm{mL}^2 + \frac{1}{12} \dot{\mathrm{q}}_2 \mathrm{mL}^2$$

$$\frac{\mathrm{d}}{\mathrm{dt}} (\frac{\partial \mathrm{T}}{\partial \dot{\mathrm{q}}_\mathrm{i}}) = \frac{17}{12} \ddot{\mathrm{q}}_1 \mathrm{mL}^2 + (\frac{\cos(\mathrm{q}_1 - \mathrm{q}_2)}{2} (\dot{\mathrm{q}}_2))' \mathrm{mL}^2 + \frac{1}{12} \ddot{\mathrm{q}}_2 \mathrm{mL}^2$$

But the derivative of the parenthesis yields :

$$(\frac{\cos(q_1-q_2)}{2}(\dot{q}_2))' = \frac{1}{2}(\ddot{q}_2\cos(q_1-q_2)-\dot{q}_2(\dot{q}_1-\dot{q}_2)\sin(q_1-q_2))$$

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_i}) = \frac{17}{12}\ddot{q}_1 mL^2 + \frac{\cos(q_1-q_2)}{2}(\ddot{q}_2)mL^2 - \frac{1}{2}mL^2\dot{q}_2(\dot{q}_1-\dot{q}_2)\sin(q_1-q_2) + \frac{mL^2}{12}\ddot{q}_2$$

$$\frac{\partial T}{\partial q_i} = -\frac{1}{2}mL^2\dot{q}_1\dot{q}_2\sin(q_1-q_2)$$

The potential energy function is:

$$V = mg(\frac{L}{2}-\frac{L}{2}\cos q_1) + mg(\frac{3}{2}L - L\cos q_1 - \frac{1}{2}mgL\cos q_2)$$

$$\frac{\partial V}{\partial q_1} = \frac{mgL}{2}\sin q_1 + mgL\sin q_1 = \frac{3mgL}{2}\sin q_1$$

Substitution at the original equation gives :

$$\frac{17}{12}\ddot{q}_1 mL^2 + \frac{\cos(q_1-q_2)}{2}(\ddot{q}_2)mL^2 - \frac{1}{2}mL^2\dot{q}_2(\dot{q}_1-\dot{q}_2)\sin(q_1-q_2) + \frac{mL^2}{12}\ddot{q}_2$$

$$+\frac{1}{2}mL^2\dot{q}_1\dot{q}_2\sin(q_1-q_2) + \frac{3}{2}mgL\sin q_1 = 0$$

$$\frac{17}{12}\ddot{q}_1 mL^2 + \frac{(\cos(q_1-q_2)+\frac{1}{6})}{2}(\ddot{q}_2)mL^2 + \frac{1}{2}mL^2\dot{q}_2{}^2\sin(q_1-q_2) + \frac{3}{2}mgL\sin q_1 = 0$$

Dividing by the common factor ($mL^2/2$) we get :

$$\frac{17}{6}\ddot{q}_1 + \ddot{q}_2(\cos(q_1-q_2)+\frac{1}{6}) + \dot{q}_2{}^2\sin(q_1-q_2) + \frac{3g}{L}\sin q_1 = 0$$

For the second generalized coordinate Lagrange's equation yields :

$$\frac{\partial T}{\partial \dot{q}_2} = \frac{1}{6}(2\dot{q}_2)mL^2 + \frac{\cos(q_1-q_2)}{2}(\dot{q}_1)mL^2 + \frac{1}{12}\dot{q}_1 mL^2$$

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_i}) = \frac{1}{3}\ddot{q}_2 mL^2 + (\frac{\cos(q_1-q_2)}{2}(\dot{q}_2))'mL^2 + \frac{1}{12}\ddot{q}_1 mL^2$$

But the derivative of the parenthesis yields :

$$(\frac{\cos(q_1-q_2)}{2}(q_2))' = \frac{1}{2}(\dot{q}_1\cos(q_1-q_2)-\dot{q}_1(\dot{q}_1-\dot{q}_2)\sin(q_1-q_2))$$

The derivative of the potential energy function is:

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_2}) = \frac{1}{3}\ddot{q}_2 mL^2 + \frac{\cos(q_1-q_2)}{2}(\ddot{q}_1)mL^2 - \frac{1}{2}mL^2\dot{q}_1(\dot{q}_1-\dot{q}_2)\sin(q_1-q_2) + \frac{mL^2}{12}\ddot{q}_1$$

$$\frac{\partial T}{\partial q_i} = \frac{1}{2}mL^2\dot{q}_1\dot{q}_2\sin(q_1-q_2)$$

$$\frac{\partial V}{\partial q_2} = \frac{mgL}{2}\sin q_2$$

Substitution in the original equation gives :

$$\frac{1}{3}\ddot{q}_2 mL^2 + \frac{\cos(q_1 - q_2)}{2}(\ddot{q}_1)mL^2 - \frac{1}{2}mL^2\dot{q}_1(\dot{q}_1 - \dot{q}_2)\sin(q_1 - q_2) + \frac{mL^2}{12}\ddot{q}_1$$

$$-\frac{1}{2}mL^2\dot{q}_1\dot{q}_2\sin(q_1 - q_2) + \frac{1}{2}mgL\sin q_2 = 0$$

$$\frac{1}{3}\ddot{q}_2 mL^2 + \frac{(\cos(q_1 - q_2) + \frac{1}{6})}{2}(\ddot{q}_1)mL^2 - \frac{1}{2}mL^2\dot{q}_1^{\,2}\sin(q_1 - q_2) + \frac{1}{2}mgL\sin q_2 = 0$$

Dividing by the common factor (mL^2) we get :

$$\frac{1}{3}\ddot{q}_2 + \ddot{q}_1(\frac{1}{2}\cos(q_1 - q_2) + \frac{1}{12}) - \dot{q}_1^{\,2}\sin(q_1 - q_2) + \frac{g}{2L}\sin q_2 = 0$$

So our system of deferential equations becomes:

$$\frac{17}{6}\ddot{q}_1 + \ddot{q}_2(\cos(q_1 - q_2) + \frac{1}{6}) + \dot{q}_2^{\,2}\sin(q_1 - q_2) + \frac{3g}{L}\sin q_1 = 0$$

$$\frac{1}{3}\ddot{q}_2 + \ddot{q}_1(\frac{1}{2}\cos(q_1 - q_2) + \frac{1}{12}) - \dot{q}_1^{\,2}\sin(q_1 - q_2) + \frac{g}{2L}\sin q_2 = 0$$

Now if we assume small amplitudes of vibration the equations can be linearized about the equilibrium position, we take;

$\sin(q_1) = q_1$

$\sin(q_2) = q_2$

$\sin(q_1 - q_2) = q_1 - q_2$

$\cos(q_1 - q_2) = 1.0$

and;

$$\dot{q}_1^{\,2} = 0$$

The system of differential equations becomes :

$$\frac{17}{6}\ddot{q}_1 + \ddot{q}_2(1 + \frac{1}{6}) + \dot{q}_2^{\,2}\sin(q_1 - q_2) + \frac{3g}{L}q_1 = 0$$

$$\frac{1}{3}\ddot{q}_2 + \ddot{q}_1(\frac{1}{2} + \frac{1}{12}) - \dot{q}_1^{\,2}\sin(q_1 - q_2) + \frac{g}{2L}\sin q_2 = 0$$

Which in matrix form can be written as :

$$\begin{bmatrix} \frac{17}{6} & \frac{7}{6} \\ \frac{1}{3} & \frac{7}{12} \end{bmatrix} * \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} \frac{3g}{L} & 0 \\ 0 & \frac{g}{L} \end{bmatrix} * \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

MATLAB routine to compute the gradient of the state for the double pendulum

```
function [xd]=dpend(T,X);
be=1/12;
g=386.4;
L=12;
a=2*(2*be+5/4);
b=2*be+cos(X(1)-X(3));
c=(X(4)^2)*sin(X(1)-X(3))+3*g/L*sin(X(1));
d=2*(be+1/4);
e=g/L*sin(X(3))-(2*X(2)^2)*sin(X(1)-X(3));
A=[a b;b d];
XD=-inv(A)*[c;e];
BB=XD(1);
DD=XD(2);
AA=X(2);
CC=X(4);
xd=[AA BB CC DD]';
```

Chaotic Behavior

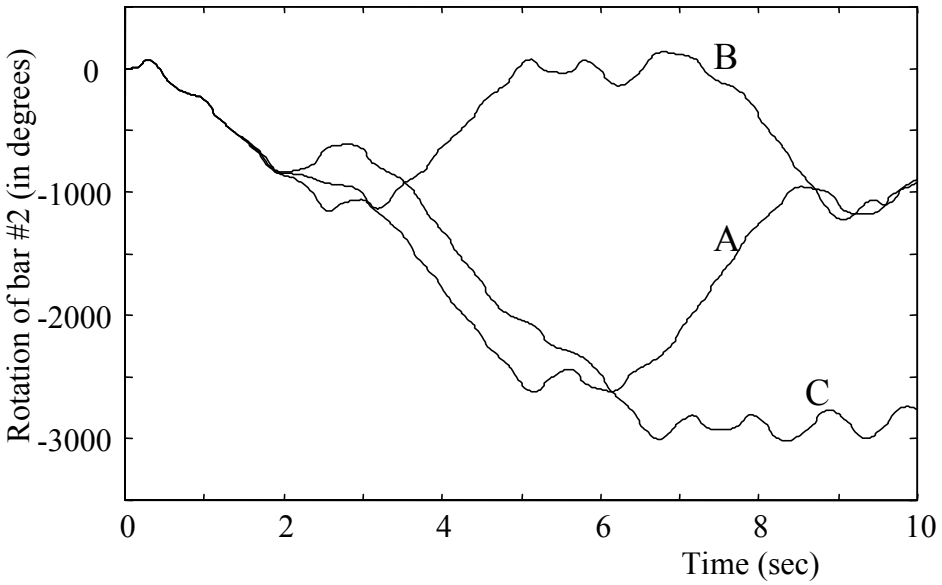
Loosely speaking, a system is chaotic when the response for nearly identical initial conditions and excitations are sharply different. Given that measurement of the state, and or the specification of a loading can only be done with finite precision, chaotic systems appear unpredictable. The double pendulum is one of the simplest systems that can display chaotic behavior. We can illustrate this by plotting the response associated with three nearly identical initial conditions. For example, we may consider releasing the pendulum from the positions:

A where $q_1 = 90.00$ and $q_2 = 0$

B where $q_1 = 90.01$ and $q_2 = 0$

and

C where $q_1 = 89.99$ and $q_2 = 0$



The results obtained for the angle of the second bar are plotted for 10 seconds in the figure.

Clearly, the time histories differ drastically for nearly identical initial conditions. If the task was to predict the rotation of bar #2 (q_2) for the initial condition $q_1 = 90$ and $q_2 = 0$ we would compute curve A. If we go out and try to perform an experiment to reproduce the results, however, we find it impossible because when we think we have 90 degrees we actually have 90.01 or something else. Predicting the response of chaotic systems is, therefore, nearly impossible -not because they are not conceptually deterministic, but because the solution is hyper-sensitive to the initial state.

Lagrange multipliers

The method of Lagrange multipliers is a technique that finds application in the following class of problems: a) computation of the values of the coordinates that make a function stationary when some of the coordinates are related by constraints.

b) evaluation of the conditions that must be satisfied to extremize a functional in the presence of constraints.

Our interest in Lagrange multipliers lies in the evaluation of the equations of motion for cases where the expressions for the kinetic and potential energy functions are obtained using more coordinates than the number of DOF and we don't wish to modify the expressions to enforce the constraints. For example, a pendulum formed by a rigid bar has a single DOF but we may wish to write the expressions for T and V (to be used in deriving the equation of motion) in terms of the coordinates x and y of the center of mass. The technique of Lagrange multipliers allows us to keep both coordinates if we don't want to modify T and V to enforce the relationship between them.

Stationarity of a Function Under Constraints.

To aid in understanding the use of Lagrange multipliers in the case of functionals, it is appropriate to introduce the technique by considering first the matter of finding the extremum of a function of several variables when constraints are present. Assume that we have a function of several variables *F*, namely:

$$F = F(x_1, x_2, \ldots x_n) \tag{1}$$

the expression for the differential of the function is

$$dF = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} dx_i \tag{2}$$

at a stationary point the differential equals zero and, provided all the coordinates are independent, one concludes that the condition in eq.2 can only be satisfied provided;

$$\frac{\partial F}{\partial x_i} = 0 \qquad \text{for each i} \tag{3}$$

The condition in (3) gives *n* equations in *n* unknowns which are satisfied at the coordinates for which *F* is stationary. To know when a particular stationary point is associated with a maximum, a minimum of a saddle point one calculates the eigenvalues of the matrix [A], given by;

$$[A] = \begin{bmatrix} F_{x1,x1} & F_{x1,x2} & F_{x1,x3}\ etc \\ F_{x2,x1} & F_{x2,x2} & F_{x2,x3} \\ F_{x3,x1} & F_{x3,x2} & F_{x3,x3} \end{bmatrix} \quad (4)$$

where the subscripts stand for derivatives with respect to the variable noted. When all the eigenvalues are positive the point considered is a minimum, when they are all negative the point is maximum and if some are positive and some are negative it is a saddle point. If the determinant of the matrix is zero further computations are necessary but we leave that special case aside for the time being.

Constraints

Assume now that the variables x_i are not all independent but that there are several equations of constraint that relate them. We express the kth constraint equation as;

$$G_k(x_1, x_2, \ldots x_n) = c_k \quad (5)$$

where c_k is a constant. Since the differential of a constant is zero we can write;

$$dG_k = \sum_{i=1}^{n} \frac{\partial G_k}{\partial x_i} dx_i = 0 \quad (6)$$

Multiplying each one of the eq.6 by a constant λ_k, known as the Lagrange multiplier, and adding the result to eq.2 one gets;

$$dF = \sum_{i=1}^{n} \left[\frac{\partial F}{\partial x_i} + \sum_{k=1}^{nc} \lambda_k \frac{\partial G_k}{\partial x_i} \right] dx_i \quad (7)$$

where *nc* = number of constraint equations. As noted previously, when the coordinates x_i are independent we can argue that, at the extremum, the coefficients of dx_i must vanish since they can all be selected arbitrarily. In other words, we can always choose *n-1* values of dx_i as zero and argue that the coefficient of the nth term must be zero. The foregoing argument, however, is not valid when the coordinates are related by constraints because if we select *n*-1-*nc* values of dx_i as

zero only one of the remaining ones is arbitrary (given that *nc* are fixed by the constraints). Needless to say, if there is more than one term in a summation that must be zero one can not argue that each individual term must be zero. One can argue, however, that the *nc* values of λ_k in eq.7 can be selected to ensure that, at the extremum, the coefficients in the brackets for the *nc* dependent terms are zero. With this reasoning we reach the interesting conclusion that <u>in fact</u> the conditions that extremize *F* are obtained by setting each of the terms in the bracket equal to zero.

Note that the requirement that the bracket in eq.7 be zero for each i gives only n equations so its clear that more information is needed to solve for the n+nc unknowns. The additional equations needed are simply the constraint equations given in (5). Although it may appear at first glance that the use of eq.5 is redundant, since it was already to arrive at eq.(7), note that what was used was only the total differential, which does not include information on the constant c_k.

The example presented next illustrates the application of the expressions derived.

<u>Example:</u>

Consider the problem of finding the point on the plane 2x-y-z which is closest to the origin. While the function *F* that we wish to extremize is the distance, it is more convenient to work with the square of the distance (to avoid the square root). The function to be extremized can thus be taken as:

$$F(x,y,z,) = x^2 + y^2 + z^2$$

and the constraint is;

$$2x - y + z = 3$$

We compute the differential of the constraint and get

$$2\,dx - dy + dz = 0$$

so, from eq.7 we have

$$(2x + 2\lambda\ ) = 0$$

$(2y - \lambda) = 0$

$(2z + \lambda) = 0$

and the 4th equation is simply the constraint equation.

Solving these equations one gets:

x = 1
y = -0.5
z = 0.5
λ = -1

Lagrange's Equations under Constraints

In the derivation of Lagrange's equations we started from Hamilton's principle and assumed that the kinetic and potential energy terms, as well as the work of the non-conservative forces, could be written in terms of a set of generalized coordinates. The expression that was obtained is;

$$\int_{t1}^{t2} \sum_{i=1}^{n} \left[\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} - Q_i \right) \delta q_i \right] dt = 0 \qquad (8)$$

When the coordinates q_i are independent one can argue that all except one of the variations can be taken as zero and thus the only way to satisfy eq.8 is for the term inside the parenthesis to be zero for all i. If there are constraints, however, we can't select the variations independently and the previous argument no longer holds. In a way that parallels entirely the treatment presented previously for functions, however, we can use the technique of Lagrange multipliers to avoid having to explicitly introduce the equations of constraint to reduce the number of coordinates. The only distinction, as we show in the following, is that the Lagrange multipliers are in this case not constants but functions of time.

The constraints are again given by eq.5, except that for consistency we replace the x's by q's. The variation of the constraint equation is given by;

$$\delta G_k = \sum_{i=1}^{n} \frac{\partial G_k}{\partial q_i} \delta q_i = 0 \tag{9}$$

Assume now that we multiply eq.9 by an arbitrary function of time $\lambda_k(t)$, which we identify as the Lagrange multiplier. Integrating from t1 to t2 one gets;

$$\int_{t1}^{t2} \sum_{i=1}^{n} \left(\lambda_k(t) \frac{\partial G_k}{\partial q_i} \delta q_i \right) dt = 0 \tag{9}$$

Adding eq.10 to eq.8 and recognizing that in general there are *nc* constraints one gets;

$$\int_{t1}^{t2} \sum_{i=1}^{n} \left[\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} - Q_i + \sum_{k=1}^{nc} \lambda_k(t) \frac{\partial G_k}{\partial q_i} \right) \delta q_i \right] dt = 0 \tag{10}$$

Again, if there are *nc* constraints we can only choose n-1-nc variations to be independent. Following the same line of reasoning as before, however, we can require that the arbitrary functions $\lambda_k(t)$ be selected in such a way that, at the extremum, the term inside the bracket

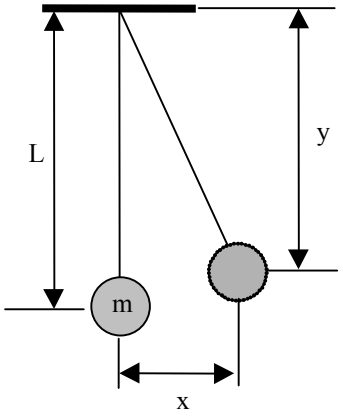
vanishes for *nc* of the coordinates. We conclude, therefore, that with this condition the variations are arbitrary and thus the equations of motion under constraints are:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} + \sum_{k=1}^{nc} \lambda_k(t) \frac{\partial G_k}{\partial q_i} = Q_i \tag{11}$$

As before, the equations of constraint must be appended to solve for the $n + nc$ unknowns.

Example

A pendulum made with a massless rigid bar and a point mass attached at the tip is shown in figure. Obtain the equations of motion in terms of the coordinates q_1 and q_2 – use the Lagrange multipliers technique to consider the constraint. Solve the equations for free vibration when the pendulum is released from 45 degrees.



The equations of kinetic and potential energy are:

$$T = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) \tag{1}$$

$$V = mg(L - y) \tag{2}$$

and the constrain equation is:

$$x^2 + y^2 = L^2 \tag{3}$$

For notational convenience we set

$$x = q_1$$

and

$$y = q_2$$

and write the expressions as

$$T = \frac{1}{2} m\left(\dot{q}_1{}^2 + \dot{q}_2{}^2\right) \tag{4}$$

$$V = mg(L - q_2) \tag{5}$$

$$q_1{}^2 + q_2{}^2 = L^2 \tag{6}$$

The terms needed in Lagrange's equation with constraints are;

$$\frac{\partial T}{\partial \dot{q}_1} = m\dot{q}_1 \tag{7}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = m\ddot{q}_1 \tag{8}$$

$$\frac{\partial T}{\partial \dot{q}_2} = m\dot{q}_2 \tag{9}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) = m\ddot{q}_2 \tag{10}$$

$$\frac{\partial T}{\partial q_1} = 0 \tag{11}$$

$$\frac{\partial T}{\partial q_2} = 0 \tag{12}$$

$$\frac{\partial G}{\partial q_1} = 2q_1 \tag{13}$$

$$\frac{\partial G}{\partial q_2} = 2q_2 \tag{14}$$

$$\frac{\partial V}{\partial q_1} = 0 \tag{15}$$

$$\frac{\partial V}{\partial q_2} = -mg \tag{16}$$

substituting in Lagrange's equation with constraints (eq.11 of previous section) we obtain:

$$m\ddot{q}_1 + \lambda 2q_1 = 0 \tag{17}$$

$$m\ddot{q}_2 - mg + \lambda 2q_2 = 0 \tag{18}$$

$$q_1{}^2 + q_2{}^2 = L^2 \tag{19}$$

To solve this system by transformation to first order we need to eliminate λ (since no derivatives of this variable appear). Multiplying eq.17 by q_2/q_1 and adding the result to eq.18 we get;

$$m\ddot{q}_1 \frac{q_2}{q_1} - m\ddot{q}_2 = -mg \tag{20}$$

or,

$$m\ddot{q}_1 q_2 - m\ddot{q}_2 q_1 = -mgq_1 \tag{21}$$

Adding the constraint equation (eq.19) to eq.21 we get;

$$m\ddot{q}_1 q_2 - m\ddot{q}_2 q_1 + mgq_1 + q_1^2 + q_2^2 = L^2 \tag{22}$$

We convert eq.22 to first order form by setting:

$$x_1 = q_1 \tag{23}$$

$$x_2 = \dot{q}_1 \tag{24}$$

$$x_3 = q_2 \tag{25}$$

$$x_4 = \dot{q}_2 \tag{26}$$

substituting eq.23-26 into eq.22 one gets;

$$m\dot{x}_2 x_3 - m\dot{x}_4 x_1 + mgx_1 + x_1^2 + x_3^2 = L^2 \tag{27}$$

which provides a relationship between $\dot{x}_2$ and $\dot{x}_4$; we also have;

$$\dot{x}_1 = x_2 \tag{28}$$

and

$$\dot{x}_3 = x_4 \tag{29}$$

since we have four unknowns and only three equations and additional relationship is needed. While it may seem that we have used all the available information, note that the constraint equation was incorporated by adding it to eq.21 so the need to satisfy it independently has not been utilized.

Differentiating eq.19 we get;

$$2x_1\dot{x}_1 + 2x_3\dot{x}_3 = 0 \tag{30}$$

using eqs28 and 29 to replace the derivatives one gets

$$x_1 x_2 + x_3 x_4 = 0 \tag{31}$$

differentiating a second time one gets

$$\dot{x}_1 x_2 + \dot{x}_2 x_1 + \dot{x}_3 x_4 + \dot{x}_4 x_3 = 0 \tag{32}$$

finally, substituting eqs28 and 29 gives

$$x_2^2 + \dot{x}_2 x_1 + x_4^2 + \dot{x}_4 x_3 = 0 \tag{33}$$

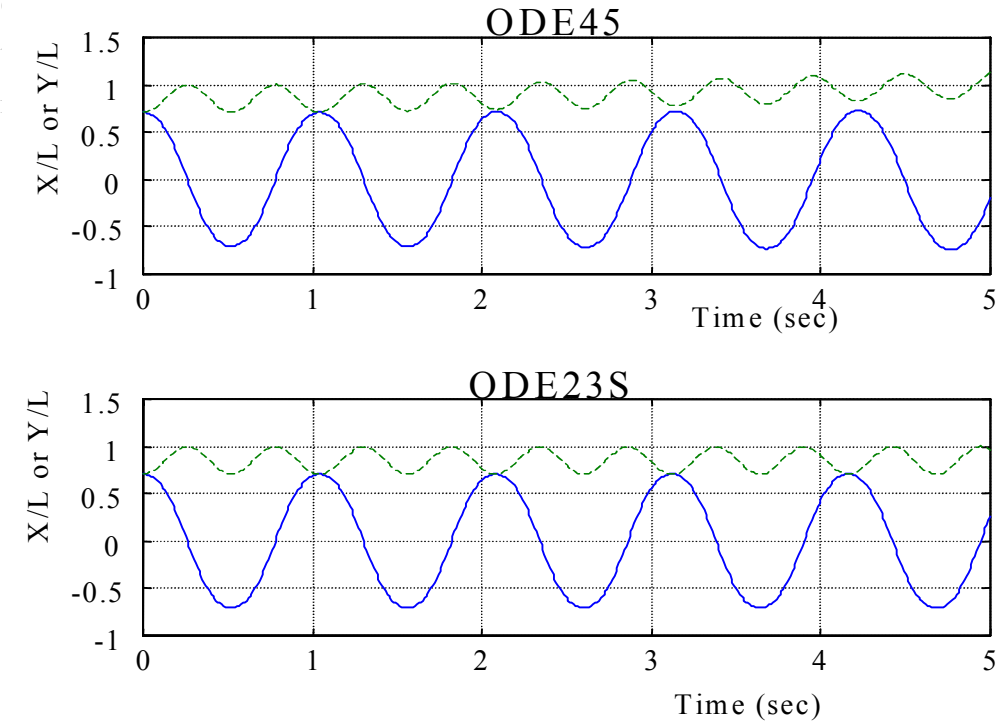
which provides the needed additional relationship. Combining eqs 27 and 33 one can solve for $\dot{x}_2$ and $\dot{x}_4$ - we have;

$$\begin{bmatrix} x_3 & -x_1 \\ x_1 & x_3 \end{bmatrix} \begin{Bmatrix} \dot{x}_2 \\ \dot{x}_4 \end{Bmatrix} = -\begin{Bmatrix} gx_1 + \dfrac{x_1^2}{m} + \dfrac{x_3^2}{m} - \dfrac{L^2}{m} \\ x_2^2 + x_4^2 \end{Bmatrix} \tag{34}$$

Eqs 28, 29 and 34 provide the derivative of the state in terms of the state.

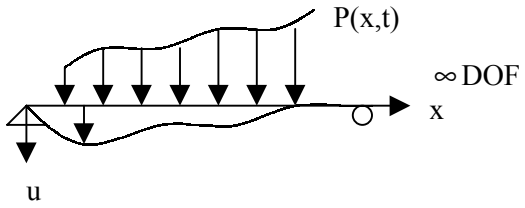
```
% Example on the use of the Lagrange Multipliers Technique.
% Routine to compute the derivative of the state from the state.
% We set the length so that the small amplitude period of
% oscillation = 1 sec.
function [xd]=lagm(T,X);
m=1;
g=386.4;
L=g/(4*pi^2);
x1d=X(2);
x3d=X(4);
A=[X(3) -X(1);X(1) X(3)];
F=-[g*X(1)+(X(1)^2+X(3)^2-L^2)/m;X(2)^2+X(4)^2];
XX=inv(A)*F;
xd=[x1d XX(1) x3d XX(2)]';
```

The solution for the x and y histories of the motion are shown in the figure. As noted, the top is using ODE 45 and the lower one corresponds to ODE23S. Note that the ODE45 solution tends to



DISCRETIZATION

In the preceding sections the equations of motion for systems with a finite set of DOF were derived using Lagrange's approach. In this section we apply the procedure to systems that have an infinite number of DOF but which are treated approximately using a finite set of generalized coordinates. Consider for illustration the beam shown in the sketch;



By restricting the deformations u(x,t) to be small we know that the equations of motion will be linear so the solution can be expressed using a separation of variables. Provided the functions $\phi_i(x)$ are complete in the domain of interest ($0 \le x \le L$ in this case) we can write;

$$u(x,t) = \sum_{i=1}^{\infty} q_i(t)\phi_i(x) \tag{1}$$

where completeness of $\phi_i(x)$ means that the functions in the set can be combined linearly to define any arbitrary function that is well-behaved (at least piece-wise continuous). An approximate solution can be obtained by restricting the number of terms in (1) to a finite number *n*. Since application of Lagrange's equation will lead to *n* equations we can say that the system has been discretized. Note that this discretization is different from that obtained by physically lumping the properties at a finite number of points.

We obtain explicit the form of the matrices in the resulting linear equations as follows. The kinetic energy is given by;

$$T = \frac{1}{2}\int m(x)\dot{u}^2(x,t)dx \tag{2}$$

substituting eq.1 into eq.2 one gets;

$$T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\dot{q}_i\dot{q}_j \tag{3}$$

where

$$m_{ij} = \int m(x)\phi_i\phi_j dx \tag{4}$$

The potential energy is equal to the work stored in the structure for a given deformed configuration. Assuming that all the energy is stored in flexural deformations (which is sufficiently accurate for slender beams) and neglecting the work done by gravity, we get;

$$V = \frac{1}{2}\int EI(x)(u''(x,t))^2 dx \tag{5}$$

where we have introduced the notation $u' = \frac{du}{dx}$ and $\dot{u} = \frac{du}{dt}$

Substituting eq.1 into eq.5 one gets;

$$V = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij} q_i q_j \tag{6}$$

where

$$k_{ij} = \int EI(x)\phi_i''\phi_j'' dx \tag{7}$$

The work done by the non-conservative forces on a variation of the displacements is given by;

$$\delta W_{nc} = \int P(x,t)\delta u(x,t) dx \tag{8}$$

substitute again eq.1 into eq.8 gives

$$\delta W_{nc} = \int (P(x,t)\sum_{i=1}^{n} \delta q_i(t)\phi_i) dx \tag{9}$$

or

$$\delta W_{nc} = \left[\int P(x,t)\sum_{i=1}^{n} \phi_i dx\right]\delta q_i \tag{10}$$

Since the variation of the non-conservative generalized forces is;

$$\delta W_{nc} = \sum Q_i \delta q_i \tag{11}$$

we conclude that;

$$Q_i = \int P(x,t)\phi_i dx \tag{12}$$

The terms in Lagrange's Equation are:

$$\frac{\partial T}{\partial \dot{q}_i} = \sum_{j=1}^{n} m_{ij} q_j \tag{13}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \sum_{j=1}^{n} m_{ij} \ddot{q}_i \tag{14}$$

which can be written in matrix form as;

$$\begin{bmatrix} m_{11} & m_{12} & \dots \\ m_{21} & & \\ \vdots & & \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \vdots \end{Bmatrix}$$

So we see that $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)$ gives $[m]\{\ddot{q}\}$

$$\frac{\partial T}{\partial q_i} = 0$$

Another term in the equation is;

$$\frac{\partial V}{\partial q_i} = \sum_{j=1}^{n} k_{ij} q_j$$

which in matrix form is;

$$\begin{bmatrix} k_{11} & k_{12} & \dots \\ k_{21} & & \\ \vdots & & \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \end{Bmatrix}$$

So we see that $\frac{\partial V}{\partial q_i} \rightarrow$ gives $[k]\{q\}$

Therefore, neglecting damping

$$[m]\{\ddot{q}\}+[k]\{q\}=\{Q\}$$

where

$$m_{ij} = \int m(x)\phi_i\phi_j dx$$

$$k_{ij} = \int EI\phi_i''\phi_j''$$

and

$$Q_i = \int P(x,t)\phi_i dx$$

where we recall that;

$$u(x,t) = \sum_{i=1}^{n} q_i(t)\phi_i$$

The previous expressions can be easily generalized to include the case of lumped parameters, one gets:

$$m_{ij} = \int m(x)\phi_i\phi_j dx + \sum m_k\phi_i^k\phi_j^k + \sum J_k\phi_i'^k\phi_j'^k$$

$$k_{ij} = \int EI\phi_i''\phi_j''dx + \sum k_k\phi_i^k\phi_j^k + \sum k_{r_k}\phi_i'^k\phi_j'^k$$

$$Q_i = \int P(x,t)\phi_i dx + \sum P_k\phi_i^k + \sum m_k\phi_i'^k$$

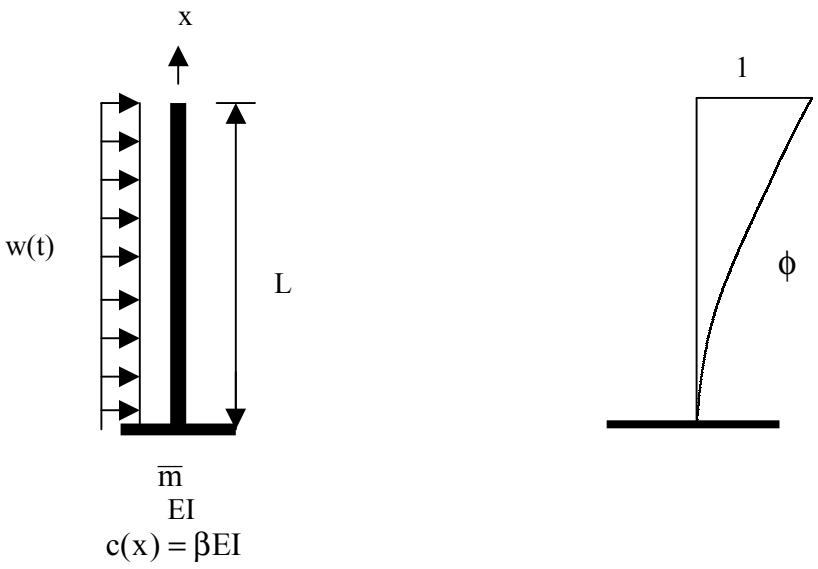
and

$$c_{ij} = \int c(x)\phi_i\phi_j dx + \sum c_k\phi_i^k\phi_j^k + \sum c_{r_k}\phi_i'^k\phi_j'^k$$

where the subscript *k* is used to indicate the location of the lumped quantity.

EXAMPLE

Consider a cantilever subjected to a dynamic load which is uniform along the height and has an arbitrary temporal variation. Obtain a SDOF using the shape function shown in the figure.



$$u(x,t) = q_1(t)\phi_1(x) \qquad \phi = \left(1 - \frac{\cos \pi x}{2L}\right)$$

Performing the integrations we get:

$$m_{11} = \int_0^L \bar{m}\left(1 - \frac{\cos \pi x}{2L}\right)^2 dx = \bar{m}\frac{L(3\pi - 8)}{2\pi}$$

$$k_{11} = \int_0^L EI\left(\frac{\pi}{2L}\right)^4 \left(\frac{\cos \pi x}{2L}\right)^2 dx = \frac{\pi^4 EI}{32L^3}$$

$$c_{11} = \beta EI \frac{L(3\pi - 8)}{2\pi} \qquad \text{(by inspection of } m_{11})$$

$$Q_i = \int w(t)\left(1 - \frac{\cos \pi x}{2L}\right) dx = w(t)L\left(\frac{\pi - 2}{\pi}\right)$$

therefore,

$$\frac{(3\pi-8)}{2\pi}\overline{m}L\ddot{q}_1+\beta EI\frac{L(3\pi-8)}{2\pi}\dot{q}_1+\frac{\pi^4 EI}{32L^3}q_1=w(t)\frac{L}{\pi}(\pi-2)$$

or, dividing by the mass term

$$\ddot{q}_1+\frac{\beta EI}{\overline{m}}\dot{q}_1+\frac{\pi^5 EI}{16L^4\overline{m}(3\pi-8)}q_1=\frac{2(\pi-2)}{\overline{m}(3\pi-8)}w(t)$$

Since the equation of motion for a SDOF system is;

$$\ddot{u}+2\omega\zeta\dot{u}+\omega^2 u=\frac{P(t)}{m}$$

we conclude that;

$$\omega^2=\frac{\pi^5 EI}{16L^4\overline{m}(3\pi-8)}$$

or

$$\omega=3.66\sqrt{\frac{EI}{\overline{m}L^4}}$$

It is interesting to note that the 1st mode frequency for a uniform cantilever beam is:

$$3.52\sqrt{\frac{EI}{\overline{m}L^4}}$$

the expression for the damping ratio is readily obtained as;

$$2\omega\zeta=\frac{\beta EI}{\overline{m}}$$

$$\zeta=\frac{\beta EI\sqrt{\overline{m}L^4}}{2\overline{m}(3.66)\sqrt{EI}}=\frac{\sqrt{E\,I\overline{m}\,L^4}}{7.32\overline{m}}\beta$$

or

$$\zeta=\frac{L^2\sqrt{\frac{EI}{\overline{m}}}}{7.32}\beta$$