EXTRACTION OF SECOND ORDER SYSTEM MATRICES FROM STATE-SPACE REALIZATIONS

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Introduction

The last two decades have seen the development of robust algorithms to construct minimum order state-space realizations from input/output data (Juang and Papa, 1984). When expressed in modal coordinates these realizations provide *n* pairs of complex eigenvalues and eigenvectors that can be related to the system's natural frequencies and damping ratios and to the values of the system modes at the sensor locations. Model update algorithms typically utilize identified modal parameters, instead of the physically measured response, as the targets to be matched in fitting a model to the data. Deviations in optimum model parameters for data collected at different times is often used to infer damage. An important difficulty in a model-update damage identification strategy is the fact that the formulation becomes ill-conditioned when the "free parameter" space is large in comparison to the number of constraints imposed by the available data. One way to ameliorate the problem is to delay the introduction of a class of model into the process by focusing on changes in system matrices obtained directly from the measured data. We use the term delay to emphasize that questions on the precise nature of damage are intrinsically model-related and cannot, therefore, be ultimately answered without reference to a model.

A first step in a "non-model" based identification approach is to separate the stiffness, damping and inertial properties so that changes in these characteristics can be independently examined. In many cases, of course, it is reasonable to assume that damage is restricted to changes in stiffness and thus the task is reduced to extracting the stiffness matrix (or its inverse the flexibility) from the results of the realization. In practice, most techniques focus on the flexibility because, in contrast to the stiffness, this matrix is dominated by the lower modes and these are the ones that are usually available from the modal identification. In addition, the flexibility can be computed at an arbitrary number of coordinates associated with sensor locations without the need to invoke extrapolation schemes to estimate the unmeasured coordinates. Changes in damping characteristics are seldom useful as damage indicators since they usually represent fluctuations in the fit of a viscous model to an undetermined dissipation mechanism.

The difficulties and need for approximation in the process of extracting system matrices from a state-space realization are intimately connected to: a) the number of independent sensors available b) the number of identified modes and c) the nature of the damping distribution. For the purpose of the following discussion we assume that the actual system, which in reality is continuous and has an infinite number of DOF, has been

replaced by a discrete model that we take as "exact". Taking N as the number of DOF, n as the number of identified modal pairs and m as the number of independent sensors one has the following situation:

1) $N = m = n$ - The system matrices can be extracted from the realization for both classical and non-classical damping distributions. Error derives only from approximation in the identified eigensolution.

2) $N = m > n$ - When the damping is classical the projection of the system matrices on the identified modes can be computed. If damping is not classical, however, it is not generally possible to obtain an exact set of undamped mass normalized modes and one is forced to accept some degree of approximation beyond that resulting from the fact that $n < N$ (Ibrahim and Sestieri 1995). The simplest and most widely used approach is to approximate the undamped modes by rotating the complex modes to minimize the imaginary part. This paper presents an alternative approach that extracts an estimate of the flexibility matrix without explicit computation of the mass normalized undamped mode shapes.

3) $N > m$ – If damping is classical one can calculate the contribution of the identified modes to the flexibility associated with the sensor locations. Mass and damping matrices of order m can also be computed but their potential usefulness in locating damage is not evident. If damping is not classical it appears one has to resort to approximate techniques to estimate undamped mass normalized modes (Alvin, Peterson and Park 1997).

The paper is organized as follows: transformation of an arbitrary realization to the basis of the available sensors is briefly reviewed. Extraction of the system matrices for the case $N = m = n$ is presented next (Alg1). This is followed by an approach for $N = m, >n$ which uses estimates of the undamped modes obtained by rotation of the complex ones (Alg2). A technique that yields an estimate of the flexibility without explicit reference to mass normalized undamped modes (Alg3) concludes the theoretical section. A numerical example illustrates the gains in accuracy that can be attained with Alg3 when modal complexity is significant and $m>n$. The case $N>m$ is not discussed in the paper.

Analytical Framework
The first task in the extraction of system matrices is to transform the realization results to a displacement-velocity (D-V) basis. The results of a realization, after transformation to continuous time, provide the input-output map in eq.1.

$$\dot{x} = A\,x + B\,u \qquad\qquad y = Cx + Gu \qquad\qquad (1a,b)$$

where x is the state vector, y is the measured output, u is the input, and the quadruple [A, B, C, G] are the matrices of the realization. By introducing a change of basis and noting that the output vector is invariant one can shown that;

$$C_1\,\Phi = C_z\,\Psi \qquad\qquad (2)$$

where, Φ and Ψ are eigenvectors of the system matrix A for two arbitrary state vectors and C_1 and C_z are the associated C matrices. Requiring that the D-V state vector be the one associated with C_1 one can write;

$$\begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix}[\Phi] = \begin{bmatrix} \Phi_d \\ \Phi_d\Lambda \end{bmatrix}[\Lambda] \tag{3}$$

where M, D and K are the mass, damping and stiffness matrices, Φ_d stands for the displacement partition of the eigenvector matrix and Λ is the eigenvalue matrix (which is unaffected by the basis selection). It can be easily shown that for the D-V basis (Juang 1994);

$$C_1 = [C_d - C_a M^{-1}K \quad C_v - C_a M^{-1}D] \tag{4}$$

where C_d, C_v, and C_a are matrices connecting the output vector y to the physical coordinates. Combining eq.2 with the second partition of eq.3 and then with eq.4 one can show that;

$$\Phi_d = C_z \psi \Lambda^{-p}, \qquad \text{or} \qquad \Phi = \begin{bmatrix} C_z\psi[\Lambda^{-p}] \\ C_z\psi[\Lambda^{-p+1}] \end{bmatrix} \tag{5,6}$$

where $p = 0, 1$ or 2 for displacement, velocity or acceleration sensing respectively. Eq.6 allows the computation of the D-V eigenvectors from those of an arbitrary realization.

Algorithm 1 - $N = m = n$
The system matrices can be extracted as follows: The inverse of the state matrix for the D-V state vector is;

$$A^{-1} = \begin{bmatrix} -K^{-1}D & -K^{-1}M \\ 0 & I \end{bmatrix} \qquad \text{or} \qquad A^{-1} = \Phi\Lambda^{-1}\Phi^{-1} \tag{7,8}$$

taking the upper right partition from eq.8 and solving the associated eigenvalue problem one gets arbitrarily normalized undamped mode shapes ϕ.

Mass Normalization
The relationship between the load influence matrix for the D-V basis and that for an arbitrary realization is;

$$B_1 = \Phi\psi^{-1}B_z \qquad \text{where} \qquad [B_1] = \begin{bmatrix} 0 \\ M^{-1}b_2 \end{bmatrix} \tag{9,10}$$

and b_2 is an $N \times r$ matrix in which column j describes the spatial distribution of the jth input and r = the number of inputs. Combining eqs.9 and 10 one gets;

$$R = M^{-1} b_2 \qquad \text{where} \qquad R = \Phi_d \Lambda \psi^{-1} B_z \qquad (11,12)$$

We define $d = \phi^T M \phi$ and note that eq.11 can be written as,

$$R = \phi \ d^{-1} \phi^T b_2 \qquad (13)$$

Equality of the *jth* columns in (13) requires that;

$$R_j = G_j di \qquad \text{where} \qquad G_j = \left[\phi_1 \phi_1^T b_{2j}, \phi_2 \phi_2^T b_{2j}, ..\phi_n \phi_n^T b_{2j}\right] \qquad (14,15)$$

Eq.14 contains *dof* equations that can be solved for the reciprocals of the coefficients in the diagonal matrix d *(di)*. For multiple input cases the results depend on the particular column of R chosen (if noise is considered) and is best to proceed with a least square solution that accounts for all the available information. One gets;

$$di = (\tilde{G}^T \tilde{G})^{-1} \tilde{G}^T \ \tilde{R} \qquad (16)$$

where $\tilde{R} = [R_1 R_2 ...R_r]^T$ and $\tilde{G} = [G_1 G_2 ..G_r]^T$. Once the undamped mass normalized modes φ are computed the system matrices follow as;

$$M = \left[\varphi \varphi^T\right]^{-1} \qquad K = \left[\varphi \omega^{-2} \varphi^T\right]^{-1} \qquad D = \left[\varphi (2\omega \xi)^{-1} \varphi^T\right]^{-1} \qquad (17,18,19)$$

Algorithm 2- $N = m > n$.
When the identified basis is not complete the system matrix in D-V form can not be formed and used to obtain the undamped modes. Since it is not generally possible to obtain real modes by rotating the identified complex ones, some approximate technique to estimate the undamped modes from the truncated available basis is necessary. Of the various procedures that have been proposed the simplest and most widely used one is to minimize the imaginary components by rotation and to take the undamped modes as either the amplitude times the sign of the real part or as the real component. We choose this alternative (using the amplitude times the sign of the real part) for Alg2.

While the mass normalization described in the previous section can be applied to the approximated undamped modes, a modified approach that improves accuracy by accounting for the effect of the rotation in eq.12 has been developed. Due to space limitations only the resulting expressions are presented next. In the modified approach the modal masses for the un-normalized modes, *d,* are given by;

$$d = (\tilde{Y}^T \tilde{Y})^{-1} \tilde{Y}^T \ \tilde{F} \qquad (20)$$

where
$$Y = [I \quad I] S \Lambda \psi^{-1} B_z, \quad \Phi_d S^{-1} \approx \phi [I \quad I] \text{ and } F = \phi^T b_2 \qquad (21,22,23)$$

and $\tilde{Y} = [diag(Y_1)\,diag(Y_2)...diag(Y_r)]^T$ and $\tilde{F} = [F_1 \ F_2 \ ... F_r]^T$. Once the mode shapes are mass normalized the modal contribution to the inverse of the system matrices are obtained from eqs.17-19. In these equations the estimates of the undamped natural frequencies are extracted from the identified complex eigenvalues in the usual way.

Algorithm 3 - $N = m > n$
An undesirable feature of the previous approach is that error depends on the accuracy with which each of the undamped modes is identified. An alternative that may reduce the sensitivity to the complexity of the individual modes is to operate with the available truncated basis as a unit. Assume that the inverse of the system matrix is computed from eq.8 by replacing Φ^{-1} with the Moore-Penrose pseudo-inverse of the identified modes. Defining the upper right partition of this matrix as $\tilde{Q}_2$ one gets (assuming classical damping);

$$\tilde{Q}_2 = -(K_1^{-1}M_1) + \mathrm{E} \tag{24}$$

where E is the error resulting from the use of the pseudo-inverse and the subscript 1 refers to the contribution from the identified modes. Post-multiplying eq.24 by the inverse of M_1 and noting that the error E lies in the null space of the identified modes one gets;

$$\tilde{Q}_2 M_1^{-1} = -K_1^{-1} \tag{25}$$

Imposing symmetry on M_1^{-1} and K_1^{-1} gives;

$$M_1^{-1}\tilde{Q}_2^T - \tilde{Q}_2 M_1^{-1} = 0 \tag{26}$$

which provides $0.5(n^2 - n)$ independent equations. The columns of M_1^{-1} associated with input locations can be obtained directly from Eq.11 and the information combined with eq.26 to solve for the remaining coefficients. Having obtained M_1^{-1} the flexibility associated with the identified modes follows from eq.26. While the approach has been derived on the assumption of classical damping, one expects error from modal complexity to be smaller than in Alg2 because the truncated basis is used as a unit without individual reference to undamped modal contributions.

Numerical example
A fixed-fixed beam with 8 lumped masses and two damping distributions is examined. In the 1st case damping is mass proportional with 5% of critical in the first mode while in the second a non-classical distribution corresponding to external dampers on masses 5-8 is considered. The excitation is white noise acting on the 6th mass. The output vector contains accelerations at each one of the masses in the vertical direction. Noise is taken as white with 5% RMS of the signal of the 4th sensor for the output and 5% of each signal for the input. Identification is carried out using ERA (Juang 1994). Four complex modal pairs were identified from examination of the singular values of the Hankel matrix for the

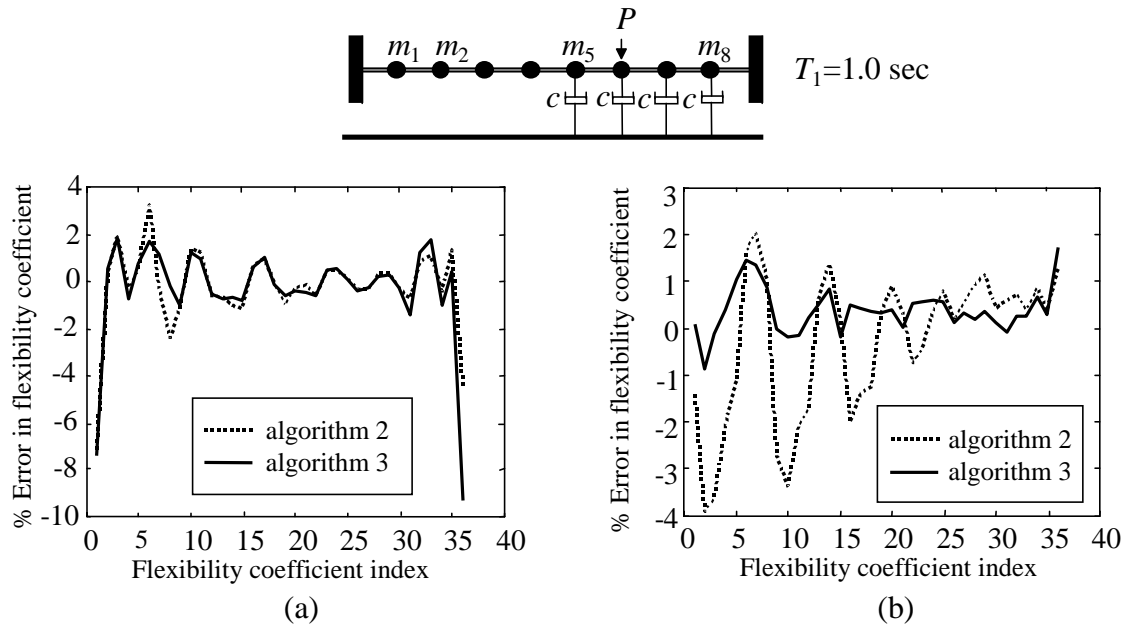classical damping distribution and six for the non-classical. The results are discussed in the concluding section.

Fig. 1 Error in flexibility coefficients (a) Classical damping (b) Non-classical damping

Concluding Remarks

Fig.1 shows the error in the flexibility coefficients obtained using algorithms 2 and 3 normalized using the largest flexibility coefficient of each particular column in the exact solution. The results confirm the expectations one has from an examination of the theory, namely: 1) both algorithms provided similar accuracy when the damping is classical and 2) algorithm #3 is superior when modal complexity is significant. We note in this regard that modal complexity may arise not only due lack of classical damping but also from noise and other spurious sources (Deblauwe and Allemang 1986). One anticipates that Alg#3 will also prove less sensitive to these spurious effects.

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