

EXTRACTING FLEXIBILITY MATRICES FROM STATE-SPACE REALIZATIONS

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SUMMARY: Robust algorithms to construct minimum order state-space realizations from input/output data have been developed in the past two decades. Extracting the flexibility matrix at the sensor coordinates from the matrices of the realizations is of interest in damage detection applications. The operation requires 1) extraction of the undamped modes at the sensor coordinates and 2) a technique to obtain constants that permit the normalization of these modes with respect to mass. This paper reviews the procedure involved in identifying the modes at the sensor locations and presents closed form equations for computing the mass normalization constants for the case when the damping distribution is classical.

KEYWORDS: Flexibility Matrices, Mass Normalization State-Space Realizations, Complex Eigenvectors, Classical Damping.

INTRODUCTION

State-space realizations can provide an accurate map between coordinates associated with measured inputs and outputs and have been proven robust under realistic levels of noise. Extraction of the properties associated with the second order dynamics from the realization results is of interest in damage localization applications. In many cases, of course, it is reasonable to assume that damage is restricted to changes in stiffness and thus, the task is reduced to extracting the stiffness matrix from the matrices of the realization. In practice, however, instead of the stiffness, the focus is generally placed on extracting its inverse, the flexibility, because this matrix is dominated by the lower modes and good approximations can be obtained even when the identified modal basis is sharply truncated. Damage localization techniques that use changes in measured flexibility at sensor locations as the basic source of information include those presented by Pandey and Biswas (1994, 1995), Toksoy and Aktan (1994), Mayes (1995), Peterson et.al. (1995) and Bernal (2000), among others.

The problem of extracting the flexibility from the matrices of a realization is equivalent to that of identifying mass normalized modes and frequencies. The first step in the process, the computation of the complex modes in standard displacement-velocity form at sensor coordinates, can be carried without difficulty independently of the number of identified modes or input/output coordinates measured. Computation of undamped modes and frequencies from the complex solution, however, is a difficult problem in the case of a general damping distribution. Much research on how to transform a truncated complex modal basis to the 'corresponding' real one has been carried out [Ibrahim and Fullekrug (1990), Ibrahim (1983), Zhang and Llalement (1987), Balmes (1997)]. It appears, however, that an exact solution may not be feasible since the truncated complex and real modes may not span

the same space (Ibrahim and Sestieri, 1995). Nevertheless, in many practical cases the magnitude of the damping, or its departure from classical, is sufficiently small that one can approximate the undamped modes, with little error, by simply rotating the displacement partition of the complex modes to minimize the imaginary part. In this paper the assumption of small complexity is used to obtain arbitrarily normalized undamped modes.

Having identified a set of undamped modes, the next step is the normalization of these modes with respect to mass. A procedure to obtain mass normalization constants from the results of the realization has been presented by Alvin and Park (1994) in connection with development of the CBSI algorithm. It appears, however, the approach is restricted to cases where the sensing is displacement or velocity. While one can always integrate the measured accelerations numerically to obtain velocities (and thus obtain a realization for velocity sensing) this alternative introduces error and is likely to run into difficulties in the presence of noise. This paper presents an approach to compute mass normalization constants that is conceptual simple and has the ability to treat acceleration sensing directly.

The paper is organized as follows. The first section reviews the transformations that are needed to extract the complex modes of the system in standard displacement-velocity form from the matrices of the realization. The derivation of expressions to compute mass normalization factors is presented in the next section. The paper concludes with a numerical example where the flexibility is synthesized using a realization for acceleration sensing and the results are compared with those obtained when velocities are first computed using numerical integration.

ANALYTICAL FRAMEWORK

The first task in the extraction of the truncated flexibility is to transform the realization results to a displacement-velocity (D-V) basis. The results of a realization, after transformation to continuous time, provide the input-output map in eq.1.

$$\dot{x} = Ax + Bu \quad (1a)$$

$$y = Cx + Gu \quad (1b)$$

where x is the state vector, y is the measured output, u is the input, and the quadruple $[A, B, C, G]$ are the matrices of the realization. By introducing a change of basis and noting that the output vector is invariant one can show that;

$$C_1 \Phi = C_z \Psi \quad (2)$$

where, Φ and Ψ are eigenvectors of the system matrix A for two arbitrary state vectors and C_1 and C_z are the associated C matrices. Requiring that the D-V state vector be the one associated with C_1 one can write;

$$\begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} [\Phi] = \begin{bmatrix} \Phi_d \\ \Phi_d \Lambda \end{bmatrix} [\Lambda] \quad (3)$$

where M , D and K are the mass, damping and stiffness matrices, Φ_d stands for the displacement partition of the eigenvector matrix and Λ is the eigenvalue matrix. It can be easily shown that for the D-V basis (Juang 1994);

$$C_1 = [C_d - C_a M^{-1} K \quad C_v - C_a M^{-1} D] \quad (4)$$

where C_d , C_v , and C_a are matrices connecting the output vector y to the physical coordinates. Combining eq.2 with the second partition of eq.3 and then with eq.4 one can show that;

$$\Phi_d = C_z \psi \Lambda^{-p} \quad (5)$$

$$\Phi = \begin{bmatrix} C_z \psi [\Lambda^{-p}] \\ C_z \psi [\Lambda^{-p+1}] \end{bmatrix} \quad (6)$$

where $p = 0, 1$ or 2 for displacement, velocity or acceleration sensing respectively. Eq.6 allows the computation of the D-V complex eigenvectors from those of an arbitrary realization.

Undamped Modes and Frequencies

As noted in the introduction, the extraction of undamped modes from a truncated set of complex modes, available at an arbitrary number of sensor locations, is a difficult problem for which an exact solution may not be feasible (Ibrahim and Sestieri 1995). Assuming that the damping is small or nearly classical, however, one can always find a diagonal matrix S for which;

$$\Phi_d S \cong \phi [I \quad I] \quad (7)$$

where the estimated undamped mode shape matrix, ϕ , is real.

The complex eigenvalues are given by;

$$\Lambda_{i,i+n} = -\xi_i \omega_i \pm i \omega_i \sqrt{1 - \xi_i^2} \quad (8)$$

where n , is the number of identified modes. Expressing eq.8 as;

$$\Lambda_i = \alpha_i + i \beta_i \quad (9)$$

it follows that one can compute ω and ξ as;

$$\omega_i = \sqrt{\alpha_i^2 + \beta_i^2} \quad (10)$$

and

$$\xi_i = \frac{-\alpha_i}{\sqrt{\alpha_i^2 + \beta_i^2}} \quad (11)$$

where ω_i and ξ_i are typically designated as the undamped natural frequency and the damping ratio. It is opportune to note that the value from eq.10 is only equal to the true undamped frequency when damping is classical, for the general case, eq.10 provides an approximation (albeit typically quite accurate) to the true undamped frequency.

Mass Normalization

Taking a Fourier transform of eqs.1a and 1b, solving for the state vector from eq.1a and substituting the result in eq.1b one gets;

$$y(\omega) = [C[I \cdot i\omega - A]^{-1} + D]u(\omega) \quad (12)$$

The Fourier transform of the output vector can be expressed in terms of that of corresponding to the displacement vector, y_D as

$$y(\omega) = (i\omega)^p y_D(\omega) \quad (13)$$

where $p = 0, 1$ and 2 for displacement, velocity or acceleration sensing.

Substituting eq.13 into eq.12 one gets;

$$y_D(\omega) = \frac{1}{(i\omega)^p} [C[I \cdot i\omega - A]^{-1}B + D]u(\omega) \quad (14)$$

With mc as the number of co-located inputs and r as total number of inputs we define the Boolean matrices $q1$ ($m \times mc$) and $q2$ ($r \times mc$) so that: $F \cdot q1$ and $u \cdot q2$ are the columns of the flexibility and the rows of the input associated with co-located sensors. Recognizing that the flexibility relates $y_D(\omega)$ to $u(\omega)$ at $\omega = 0$ one can write;

$$F q1 = \lim_{\omega \rightarrow 0} \frac{1}{(i\omega)^p} [C[I \cdot i\omega - A]^{-1}B + D]q2 \quad (15)$$

Except for the case of displacement sensing, the above limit leads to the un-determinate expression $0/0$, however, differentiating the numerator and denominator an appropriate number of times one gets;

$$F q1 = -C A^{-(p+1)} B q2 \quad (16)$$

In the simple case where $q1$ and $q2$ are both identities eq.16 gives the flexibility matrix directly. This, however, is seldom the case and one needs to use the information in eq.16 to extract the mass normalization constants. Expressing the system matrix and the flexibility in terms of their eigenvalues and eigenvectors gives;

$$\phi \left(\frac{d}{\omega} \right)^2 \phi^T q1 = -C \Psi \Lambda^{-(p+1)} \Psi^{-1} B q2 \quad (17)$$

where d is an $(n \times n)$ diagonal matrix listing the mass normalization coefficients (i.e., $\tilde{\phi}_i = \phi_i \cdot d_i$, where $\tilde{\phi}_i$ is the mass normalized mode). While Eq.17 holds in general, the important point is the fact that a mode by mode equality can be established when the damping is classical. In other words, the contribution to the flexibility matrix from mode j is the same as that given by the complex mode pair corresponding to the given eigenvalue (by the pair we mean the mode and its conjugate and the corresponding vectors in the inverse of the modal matrix). Restricting the remainder of the derivation to cases where classical damping is a reasonable assumption one can write;

$$\phi_j \left(\frac{d_j}{\omega_j} \right)^2 \phi_j^T q_1 = -2 \cdot \Re(C \psi_j \Lambda_j^{-(p+1)} \psi_j^{-T} B q_2) \quad (18)$$

Introducing the notation;

$$E_j = \phi_j \phi_j^T q_1 \quad (19a)$$

and

$$G_j = -2 \cdot \Re \left(C \psi_j \psi_j^{-T} B q_2 \frac{\omega_j^2}{\Lambda_j^3} \right) \quad (19b)$$

eq.18 can be written as;

$$E_j d_j^2 = G_j \quad (20)$$

Which is the basic expression for computing the mass normalization constants. Note that eq.20 has redundant information since d_j is a scalar and E_j and G_j are $(m \times mc)$ matrices. Provided the damping is actually classical and there is no spurious modal complexity from noise or other numerical sources, the value of d_j obtained using any of the entries in eq.20 will be (within round-off error) identical. Nevertheless, if some modal complexity is neglected in the process of computing the undamped modes then there will be a small difference depending on the entry chosen. In any case, one can always proceed using a least square approach which takes advantage of all the available information. Defining \tilde{E}_j and \tilde{G}_j as $(m \times mc \times 1)$ vectors containing columns 1,2 .. mc of E_j and G_j , the least square solution follows as;

$$d_j = \sqrt{\frac{\tilde{E}_j^T \tilde{G}_j}{\tilde{E}_j^T \tilde{E}_j}} \quad (21)$$

NUMERICAL EXAMPLE

This numerical example illustrates results for the error in the flexibility synthesized in two ways. In the first case the flexibility matrix is obtained using the matrices of an acceleration-based realization and in the second case the measured output is first transformed to velocity

so that a velocity sensing realization is obtained. Needless to say, the difference between the two results derives from the approximate nature of the numerical integration of the acceleration and, most importantly, from the magnified influence of the noise after the data is integrated. It is assumed that the acceleration output vector is recorded at a sampling rate of 200Hz. The structure selected is a planar truss with 44 bars and a total of 39 DOF, as depicted in Fig.1. All the bars are made of steel (with $E = 200$ GPa) and are taken to have an area of 64.5 cm^2 . Nine sensors recording motion in the vertical direction are located at each of the unsupported joints of the lower chord. The excitation used to generate the vibration data is random white noise applied in the vertical direction at nodes 5 and 7. Sensor noise is contemplated in the excitation and the computed response. The output noise is prescribed to have an RMS equal to 10 % of the RMS of the response measured on the sensor located at joint 2 and the input noise RMS is 5% of the excitation level. Viscous dissipation is included in the form of Rayleigh damping with a magnitude of 5% of critical in the two modes having the largest modal mass in the y-y direction.

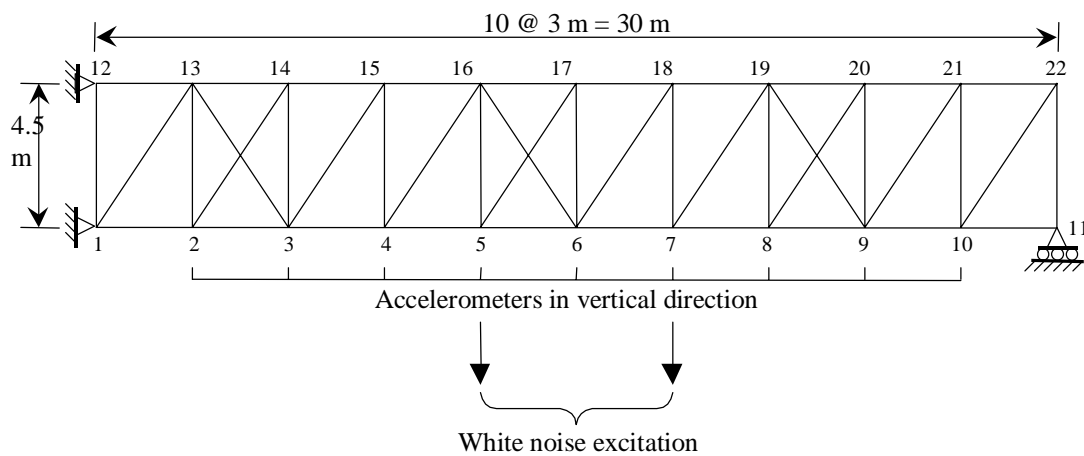


Fig.1 Structure of Numerical Example.

Modal Identification

Application of the ERA-OKID algorithm to the data identified 19 modes associated with significant singular values and having MAC indices in excess of 0.9. The periods of the first eight identified modes are compared to the exact values in Table1. The accuracy of the identification, even in the region where the modal density is high, is evident from a cursory examination. For simplicity in the comparison, we have omitted the second mode from the exact solution, which doesn't show up in the identification because it has essentially zero participating mass in the y-y direction.

Table 1. Exact and Identified Periods (sec) for the First 8 Modes

Mode	Exact	Identified	Mode	Exact	Identified
1	1.5520	1.5527	5	0.2825	0.2781
2	0.6486	0.6486	6	0.2551	0.2541
3	0.4108	0.4107	7	0.2307	0.2304
4	0.3054	0.3066	8	0.2105	0.2103

Flexibility Matrices

To isolate the effect of noise from that of the approximation introduced by the numerical integration of acceleration we have computed the flexibility matrix with acceleration data,

with and without noise, and with velocity data, with and without noise. The results are presented in Fig.2. The error in the coefficients listed in this figure represent the difference between the computed result and the exact value normalized by the largest value in the associated column of the exact matrix. In the figure, the relationship between the index and the location of the coefficient in the matrix is given by counting from the main diagonal downward, from column to column, sequentially.

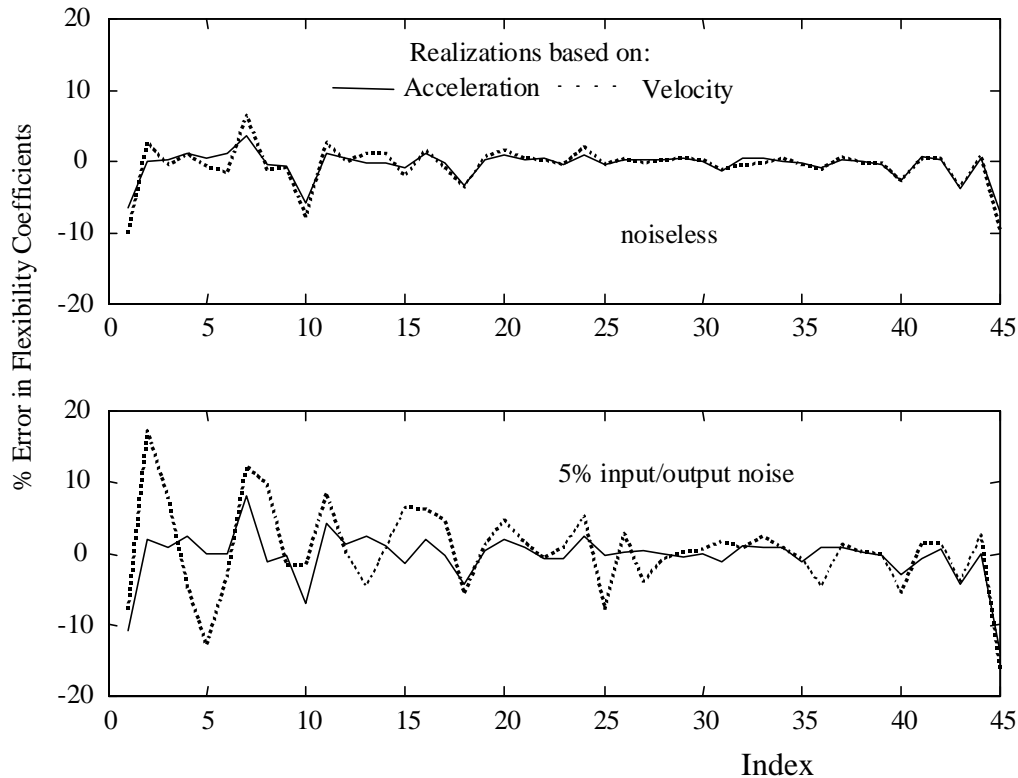


Fig.2 Error in identified flexibility coefficients (a) noiseless case (b) noise considered in input and output measurements.

An examination of Fig2.a shows that error introduced in the solution by the numerical integration of the acceleration (at the sampling rate considered and for the frequencies of the structure of this example) is not significant. As one anticipates, however, the error in the flexibility coefficients is significantly larger when noise is considered and the data is integrated prior to processing it to obtain the realization matrices.

CONCLUDING REMARKS

A review of the steps required to extract the displacement-velocity complex modes at sensor coordinates from the matrices of a state-space realization is presented. It is recalled that the identification of a set of undamped normal modes from the identified complex modes is, in the general case of an arbitrary viscous damping distribution, a difficult problem for which an exact solution may not be feasible. In particular, if the identified complex modes span a space that can not be exactly spanned by an equal number of the true undamped modes then the extraction of the undamped set appears impracticable. In connection with this issue, it is worth noting that under experimental conditions modal complexity appears, not only due to

deviations of the actual damping mechanism from classical, but also from noise and other spurious numerical effects (Deblaw and Allemang 1986).

An expression to compute constant that can be used to normalize the undamped mode shapes is derived for the case of classical damping. Central to the derivation is the fact that there is a one-to-one correspondence between the complex mode (and its corresponding vector in the inverse complex mode shape matrix) and the undamped mode shape associated with the same eigenvalue. The resulting expression for the mass normalization constant does not involve any curve fitting and, in the context of the accuracy of the measured data, is exact. Since only one co-located sensor actuator pair is needed to evaluate the mass normalization constants, redundant information is available when more than one pair is available. In the solution presented here we opted to compute the constant using a least square solution that accounts for all the available information. It may prove possible, however, to improve accuracy by using a judicious choice of which columns (and perhaps rows) of eq.20 to include in the solution.

REFERENCES

Alvin, K.F. and Park, K.C., (1994). 'Second-order structural identification procedure via state-space-based system identification', *AIAA Journal*, Vol.32, No.2, pp.397-406.

Balmes, E., (1997). 'New results on the identification of normal modes from experimental complex modes', *Mechanical Systems and Signal Processing*, Vol.11, No.2, pp.229-243.

Bernal, D., (2000). 'Damage localization using load vectors', *COST F3 Conference*, Madrid.

Deblauwe, F. and Allemang, R.J., (1986). 'A possible origin of complex modal vectors', *Proceedings of the 11th International Seminar on Modal Analysis*, Katholic University of Leuven, paper A2-3.

Doebbling, S. W., Farrar, C. R., Prime, M. B., and Shevitz, D. W., (1996). "Damage identification and health monitoring of structural and mechanical systems from changes in their vibration characteristics: A literature review." *Los Alamos Laboratory Report LA-13070-MS*.

Juang, J., (1994). Applied System Identification, *Prentice-Hall*, Englewood Cliffs, New Jersey.

Ibrahim, S.R. (1983). 'Computation of normal modes from identified complex modes', *AIAA Journal*, Vol.21, pp.446-451.

Ibrahim, S.R., and Fullekrug, U., (1990). 'Investigation into exact normalization of incomplete complex modes by decomposition transformation', *Proc. 8th International Modal Analysis Conference*, pp.205-212.

Ibrahim, S.R. and Sestieri, A. (1995). 'Existence and normalization of complex modes in post experimental use in modal analysis', *IMAC 95*, pp.483-489.

Mayes, R.L., (1995), 'An experimental algorithm for detecting damage applied to the I-40 bridge over the Rio Grande', in *Proc. 13th International Modal Analysis Conference*, pp.219-225.

Pandey, A. K. and Biswas, M., (1994). "Damage detection in structures using changes in flexibility," *J. Sound Vib.* 169(1). pp.3-17.

Pandey, A. K. and Biswas, M.,(1995). "Damage diagnosis of truss structures by estimation of flexibility change," *The International Journal of Analytical and Experimental Modal Analysis*, 10(2), pp.104-117.

Peterson, L.D., Doebling S.W., and Alvin K.F., (1995). 'Experimental determination of local structural stiffness by disassembly of measured flexibility matrices', *Proc. of 36th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference*, pp.2756-2766, AIAA-95-1090-CP.

Zhang, Q., and Lallement, G. L.,(1987). "Comparison of Normal Eigenmodes Calculation Method Based on Identified Complex Eigenmodes," *Journal of Spacecraft and Rockets*, Vol. 24, No. 1, pp. 69-73.

Toksoy, T. and Aktan, A. E.,(1994). "Bridge-condition assessment by modal flexibility." *Exp. Mech.* 34, pp.271-278.