DAMAGE LOCALIZATION IN PLATES USING DLVs

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ABSTRACT

The performance of a technique to localize damage based on the computation of load vectors that create stress fields that bypass the damaged region is investigated in the case of a plate. The Damage Locating Vectors (DLVs) are defined in sensor coordinates and are computed as the null space of the change in flexibility from the undamaged to the damaged state. The paper considers flexibility matrices computed from static measurements as well as those obtained from the modal space identified from measured vibration signals. The results confirm the anticipated difficulties for controlling the error from noise in systems with high modal density.

NOMENCLATURE

DLV	damage locating vectors = *N(DF)*
E	modulus of elasticity
F_d	damaged flexibility matrix
F_u	undamaged flexibility matrix
DF	change in flexibility matrix
$\tilde{\phi}$	mass normalized modes
ω	frequency
m	number of sensors
n	number of identified modes
δ	vector of deformations at the sensor locations
Δ	vector of deformations at all DOF
S(j)	singular value (j) of DF

INTRODUCTION

Research on various aspects associated with the use of vibration data to detect, locate and quantify damage in structures, has increased notably in the past two decades. The situation most often contemplated is that where the system considered can be treated as linear in the pre and post damaged states, making damage tantamount to a shifting of values in a set of system parameters. Linear damage characterization falls in the realm of model updating but, in contrast with the typical update problem where one searches for small adjustments to fit a base line model to measured data, in the damage detection case the adjustments need not be small. A fundamental difficulty in damage identification through a model update strategy, however, is found in the fact that the inverse problem posed, unless the number of free parameters can be made sufficiently small, is usually ill-conditioned and non-unique [Beck and Katafvgiotis (1998)][2], [Berman (19890)[3]. Methods that can extract information from the measured data to narrow the free parameter space are, therefore, of outmost practical importance.

Among methods to extract information on the spatial localization of damage, those that operate with changes in the flexibility matrix avoid the need to pair modes from the undamaged and the damage states[11, 13]. This is an attractive feature for complex systems where the pairing of modes is often difficult and is also conceptually pleasing since there is in fact no underlying one-to-one mapping between the modes at the two states. A fundamental step, once the changes in flexibility are computed, is that of converting the information into a spatial distribution of damage. While this step has traditionally been carried out using ad. hoc, system dependent procedures, a general approach with a clear theoretical underpinning has been recently introduced by Bernal [4,6]. Specifically, the method uses the null space of the change in flexibility to define vectors that have the property of inducing stress fields that bypass the damaged regions. Because of this property, these vectors are designated as Damage Locating Vectors or DLVs.

Since the DLV approach works with the null space of the change in flexibility, it is evident that its success depends on the accuracy with which the flexibility matrices can be synthesized from the measured data. In the studies carried out thus far, the DLV technique has been tested using flexibility matrices synthesized from noisy input-output data for a number of skeletal systems and it has been found to operate successfully[4,6,7]. This paper presents the results of a first examination of the technique in the case of plate structures. These systems, because of the high modal density and the relative insensitivity of the lower modal parameters to localized damage, present a particularly difficult damage localization problem.

The paper starts by reviewing the theoretical foundation of the DLV approach and discusses the particular features associated with the application to plate like structures. A section of numerical results where the DLV technique is tested, using a simply supported square plate, follow the theoretical review. In the numerical studies the flexibility is first computed from static measurements and subsequently it is synthesized from acceleration records. In all cases the damage is simulated as a reduction in the modulus of elasticity over small regions of the plate.

THEORY

The essential aspects of the DLV based approach are reviewed in this section, a more detailed treatment can be found in Bernal [6]. Consider a system that can be treated as linear in the pre and post damage states, but which is otherwise arbitrary, having damaged and undamaged flexibility matrices at *m* sensor locations given by F_D and F_U respectively. Assume there are a number of load distributions that produce identical deformations when applied to the undamaged and damaged systems. If all the distributions that satisfy this requirement are collected in the matrix L it is evident that one can write;

$$(F_D - F_U)L = DF \cdot L = 0 \tag{1}$$

Inspection of equation (1) shows that the relationship can be satisfied in two ways, either *DF = 0* or *DF* is not full rank and *L* contains the vectors that define the null space. Performing a singular value factorization one can write;

$$DF = [q_1]\begin{bmatrix} s_r & 0 \\ 0 & s_n \approx 0 \end{bmatrix}\begin{bmatrix} \tilde{q}_2^T \\ L^T \end{bmatrix} \tag{2}$$

where the ≈ 0 is introduced to emphasize that in actual applications the singular values associated with the null space will not be exactly zero due to approximations in the identified eigenproperties and possible modal truncation.

The fundamental idea in the localization approach using DLVs is that the vectors in *L*, when treated as load vectors at the sensor coordinates, lead to stress fields that bypass the damaged elements. One can appreciate this result intuitively by noting that, if a load distribution leads to zero stress in a certain element, then the change in the properties of this element will have no bearing on the computed deformations. While bypassing the damaged elements is a sufficient condition for a load vector to be contained in the *L* subspace, whether or not this condition is necessary is not apparent. Some insight into this question can be gained from the following derivation.

<u>Derivation</u>

Consider a structure for which the flexibility has been synthesized before and after damage. The incremental flexibility has been computed and the singular value decomposition has identified a certain null space *L*. Designate any one of the vectors in *L* as DLV_i. From the virtual work principle one can write;

$$DLV_i\,\delta = \int_\Omega \sigma_u\,\varepsilon_d\,dV \tag{3a}$$

and

$$DLV_i\,\delta = \int_\Omega \sigma_d\,\varepsilon_u\,dV \tag{3b}$$

Combining eqs. (3a) and (3b) and expressing the strain fields in terms of the stress field one gets;

$$\int_\Omega \sigma_u\,D_d\,\sigma_d\,dV = \int_\Omega \sigma_d D_u\,\sigma_u\,dV \tag{4}$$

where D_d and D_u are the stress to strain mapping matrices and the *u* and *d* subscripts stand for *undamaged* and *damaged* states. Consider the evaluation of eq.4 as a summation over *nf* finite size volumes ΔV. Assume the damage is such that for any one of the *nf* elemental volumes one can write;

$$D_d = \alpha\,D_u \tag{5}$$

where α is a scalar increase in flexibility ($\alpha \geq 1$ since the damage does not reduce the flexibility). Substituting (5) into (4) and recognizing that the material matrices are symmetric, one gets;

$$\sum_{i=1}^{nf} \alpha\,\sigma_u\,D_u\,\sigma_d\,\Delta V = \sum_{i=1}^{nf} \sigma_u\,D_u\,\sigma_d\,\Delta V \tag{6}$$

where the stresses are now values at appropriate points in the volumes. Inspection of eq.6 shows that the two sides are identical over those elements that are undamaged, i.e. when $\alpha = 1$. The question, therefore, is to determine the conditions under which the equality can be satisfied over the domain that contains the damaged elements. The following observations can be made:

1) If the system is statically determinate $\sigma_u = \sigma_d$. For this condition all the terms on the right side of eq.6 are positive and, since $\alpha \geq 1$, one concludes that the only way that the equation can be satisfied is if the stresses are zero over the damaged region.

2) If the system is indeterminate and there is a single damaged volume (element) it is evident that eq.6 can be satisfied only if σ_u is zero in the damaged component. Note that $\sigma_u = 0$ implies $\sigma_d = 0$ since the states are indistinguishable when the damaged elements are not stressed.

In the general case of a statically indeterminate system with multiple damaged elements, however, it is not possible to conclude from eq.6 that σ_u and σ_d must be zero in the damage region because negative terms on the right hand side cannot be discarded. Nevertheless, in spite of explicit efforts, no counter example showing a vector in *L* that does not bypass the damage region has been found thus far.

ON THE SIZE OF THE NULL SPACE OF *DF*

Consider a skeletal system having *p* DOF and *m* sensors and assume that the internal forces in all the elements are arranged in a vector *z.* Since the system is linear, it is evident that one can write;

$$Q\,\Delta = z \tag{7}$$

where Δ is a p x 1 vector listing all the DOF and *Q* is an appropriately sized matrix of influence coefficients. Furthermore, if *V* is an *m* x 1 arbitrary load vector defined in sensor coordinates we also have;

$$\Delta = G\,V \tag{8}$$

where *G* contains appropriate partitions of the full sized flexibility matrix. Combining eqs. 7 and 8 one gets;

$$R\,V = z \tag{9}$$

where, evidently *R* = *QG*. Assume that a certain number of elements have been damaged and we want the load vector *V* to behave as a DLV, i.e., we want the internal forces in all the damaged elements to be zero. Ordering the damaged elements first and writing eq.9 in partitioned form we have;

$$\begin{bmatrix} r \\ \overline{R} \end{bmatrix} \{V\} = \begin{Bmatrix} 0 \\ \overline{z} \end{Bmatrix} \tag{10}$$

where the number of rows in *r* equals the number of internal forces to be zeroed. In order to satisfy eq.10 we have;

$$V = N(r)\beta \tag{11}$$

where N(*r*) is the null space of *r* and β is an arbitrary vector of appropriate size. It is evident from eq.11 that if the number of independent rows in *r* equals or exceeds the number of sensors there will be no null space and thus no DLVs. Moreover, one can also conclude that the number of independent DLVs equals the rank deficiency of *r*.

The number of terms in *r* needed to bypass a single element equals the number of independent parameters needed to describe the stress field. Since the number of independent stresses in any finite closed region of a continuum is infinite we conclude that there are no strict DLVs in plates. One can also reach this conclusion by focusing on the ability of the loading at the sensor points to control the DOF necessary to bypass a particular damaged region. Consider for example the 2-D skeletal system depicted in fig.1a, an arbitrary isolated member has six DOF. There are three independent rigid body motions and three deformation modes. To obtain a deformation pattern that is strictly rigid body we need, therefore, at least four independent loads (sensor locations). Consider now the plate in fig.1b, if the damage is over region Ω_d and we isolate it we can see that its 'connection' to the rest of the system is a continuous line that has ∞ independent displacements. To introduce rigid body motion over the domain Ω_d one needs, therefore, an ∞ number of independent loads.

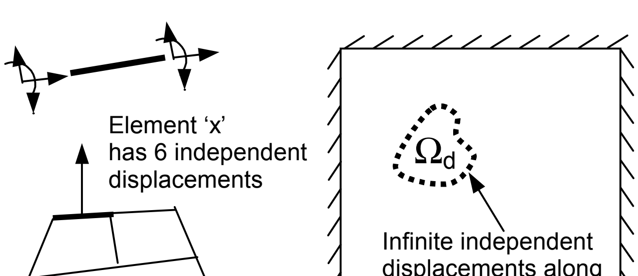


Figure (1a) Figure (1b)

Fig.1 Illustration of why plates do not have any strict DLVs

Although there are no strict DLVs in a real plate, these vectors do exist in discretized versions of the structure and this suggests that the DLV approach should still provide useful information. Consider, for example, a plate that is discretized using triangular elements having nine DOF (one translation and two rotations per node) and assume that the damage is contained within one element. Since there are three rigid body modes, the number of deformation modes equals six. Rigid body motion in one element, therefore, can be enforced if there are, at least, seven independent loads on the plate. Of course, if the region is re-meshed into more elements then the boundary of the damaged region will contain more DOF and it is no longer possible to obtain strict rigid body motion with only seven loads. It is evident, however, that if the loads can induce rigid body motion over a certain region in a reasonably accurate mesh, the stresses in this region will remain very small as the mesh is progressively refined.

In summary, while one cannot expect the change in flexibility for a plate to display a true null space, very small singular values are expected to be associated with vectors that induce near rigid body motion in certain elements. These elements will form a set that should contain the damage elements.

It is opportune to note that while the DLV vectors are intended to induce rigid body motion over the damaged elements, it is possible that elements that are not damaged may also display very small stresses. One can appreciate this mathematically by combining the second partition of eq.10 with the result in eq.11, namely;

$$\overline{R}\,N(r)\,\beta = \overline{z} \tag{12}$$

In particular, one notes that when the rows of the product $\overline{R}\,N(r)$ are zero then the entries in $\overline{z}$ are also zero and these elements cannot be separated from the damaged ones by the DLVs. Needless to say, the number of available sensors plays a critical role on the sharpness of the identified set.

MODAL TRUNCATION

As noted previously, the success of the DLV approach depends upon the ability to identify the undamaged and damaged flexibility matrices with sufficient accuracy. Since all real systems have an infinite set of modes it is evident that only truncated versions of the "real" systems can be identified in practice. Given that the sum of contributions from the identified and the truncated modal spaces give the true flexibility matrices, one can write eq.1 as;

$$(F_{D1} - F_{U1})L + (F_{D2} - F_{U2})L = 0 \qquad (13)$$

where the subscripts 1 and 2 indicate identified and truncated modes respectively. What we solve to compute the DLVs is just the first term of eq.13. One concludes, therefore, that these vectors will be accurate only if they are nearly orthogonal, not only to the identified *DF*, but also to the changes in flexibility associated with the unidentified modes.

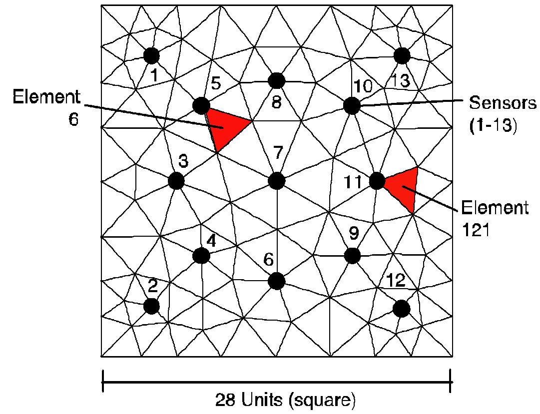
Note that because of the distributed nature of the mass, the modal density in plates is high even in the lower frequencies. The truncated modal space, therefore, may be a more significant source of error in these structures than in the skeletal systems where the DLV technique has been previously examined

NSI

From inspection of the singular value of the *DF* matrix one may conclude that there are several DLV vectors. If one decides to use more than one DLV in the localization, a procedure for combining the information given by each vector has to be established. A possible approach is simply to compute the stress fields for each DLV, add their absolute values and then normalize the results so the element with the largest 'accumulated stress' is given a value of one. The normalized stresses obtained in this fashion can then be used to rank the elements and to decide on a threshold below which elements are to be assigned to the set of potentially damaged ones. We have designated the stresses computed in this fashion as the 'normalized stress index' value or *nsi.* Further discussion on this definition, including guidelines on how to apply it in the case of structures that contain elements that are governed by different types of stress resultants, is presented in Bernal [4,6].

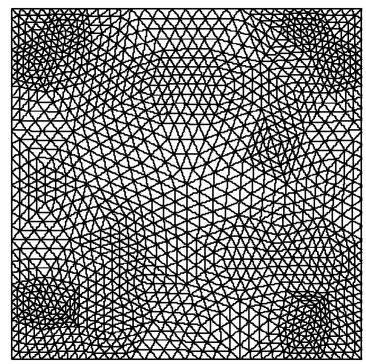
LOCALIZATION USING FLEXIBILITY MATRICES FROM STATIC MEASUREMENTS

This section presents results from numerical simulations where the flexibility matrices are assumed available from static measurements. In all cases the structure selected is a simply supported plate with dimensions 280 x 280 units, and thickness of 1 unit. The finite element model is created using triangular thin Kirchhoff plate elements with 3 DOF per node. Fig.2 illustrates two different mesh refinements for the plate and the location of the thirteen sensors assumed available. The modulus of elasticity of the plate is chosen to make the theoretical frequency of the fundamental mode equal to 2 Hz in mesh 1.



Mesh 1

damage case a – element 6, 50% reduction in E
damage case b – elements 6&121, 50% reduction in E



Mesh 2

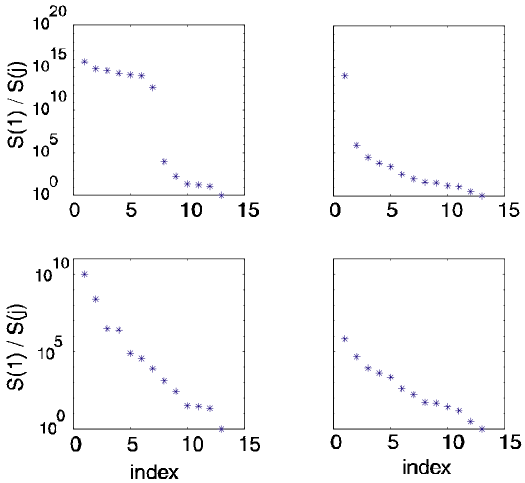
Fig.2 Plate, meshes and damage patterns considered in the numerical simulations.

Example #1

This example illustrates the performance of the DLV technique under ideal conditions. Two damage cases are considered: a) damage in one element and b) damage in two non-adjoining elements. In both cases the damaged regions coincide with single elements in the coarse mesh (see fig.2).

We begin by looking at the number of DLVs. Figure 3 plots the inverse of the normalized singular values of *DF* for the two meshes considered. As can be seen, when mesh 1 is used there are clear gaps after the 7^{th} and the 1^{st} normalized singular values for damage cases *a* and *b* respectively. These results are in agreement with the theoretical predictions, namely: 13 sensors – 6 deformation modes that need to be bypassed in damage case-a = 7DLVs; for damage case-b one has 13 sensors – 12 deformation modes = 1 DLV.

In mesh 2 the damaged regions contain 16 elements and the boundaries contain 12 nodes. The number of independent loads needed to enforce rigid body motion are therefore, 12*3 – 3 = 33 for damage case *a,* and 66 for damage case *b*. Clearly then, the 13 loads at the available sensor locations can not enforce rigid body motion and there are no DLVs in a strict sense. This result is reflected in much smaller normalized singular values and a less clear distinction of the number of terms that one would consider as belonging to the 'null space' of *DF*. Studies that have examined the issue of rank detection under noisy conditions include those by Akaike [1] and Juang [10]. In this paper, however, the number of DLVs to be used in any given case has been decided by inspection.



Damage (a) Damage (b)

mesh 1 (top), mesh 2 (bottom)

Fig.3 Normalized singular value reciprocals

The *nsi* indices for the two cases considered are depicted in Table 1. The representative stress has been taken as the Von Misses stress at the centroid of the elements. The DLV vectors are based on the most refined mesh (which is considered representative of what the 'true plate' would have yielded in the ideal situation of noiseless measurements). A cursory inspection of the results in the table shows that the damaged elements are in all cases contained in the set of elements having very low *nsi* values.

Table 1. *nsi* values for example #1

Damage Case (a)			Damage Case (b)	
0% Noise - dlv 4[&]			0% Noise - dlv 5[&]	
element	nsi		element	nsi
64	0.0081		***6**	**0.0612**
138	0.0083		122	0.0756
***6**	**0.0092**		***121**	**0.0810**
137	0.0106		148	0.0873
91	0.0140		91	0.1332
107	0.0141		62	0.1337
97	0.0147		79	0.1380
15	0.0155		42	0.1501
62	0.0157		29	0.1511
42	0.0163		85	0.1512
127	0.0166		21	0.1514

* damage elements
[&] the number in the DLV vector indicates the total number of vectors
used in the computation of the *nsi* values.

In actual applications one has to decide on a cutoff for the *nsi* value below which the elements are to be considered as belonging to the set of potentially damaged ones. It is opportune to emphasize that the DLV approach does not claim to identify the damaged elements but rather to produce a set where the damaged elements are contained. Further discrimination, if necessary, can be done in the framework of a model update strategy where the free parameter space has been reduced by the DLV localization.

Example #2
In this example we take a first look at how noise may affect the computed *nsi* values. The flexibility matrix is still obtained from static measurements but in this case we contaminate the flexibility coefficients by adding noise with a zero mean uniform probability distribution. The limits of the uniform distribution are taken as 4% and 10% of the smallest term in the theoretical flexibility matrix. The DLVs are again computed from the refined mesh but the *nsi* values are computed only for the coarse mesh. In other words, we simulate the real structure with the refined mesh but use the coarser mesh as the model to investigate the effect of the DLVs. In this case, therefore, the total 'error' has contributions from the modeling discrepancy (modeling error) and the added noise.

To mitigate the effect of the noise it was assumed that the static readings of the deflections were taken 20 times and averaged out. The results, which are summarized in Table 2, show that while the damaged element can be identified in the case with 4% noise, at the 10% level the approach fails. These results suggest that plates, as expected, have a larger sensitivity to noise than that of the skeletal systems previously considered.

Table 2 – *nsi* values for example 2.

4% Noise - dlv 2			10% Noise - dlv 3	
element	nsi		element	nsi
***6**	**0.0562**		47	0.0531
34	0.0991		97	0.1041
91	0.1013		42	0.1072
42	0.1053		84	0.1209
85	0.1115		66	0.1221
64	0.1116		.	.
97	0.1189		*el. 6 is the 33rd value*	
148	0.1206		.	.
107	0.1234		***6**	**0.1967**

* damaged element

LOCALIZATION USING FLEXIBILITY MATRICES FROM VIBRATION MEASUREMENTS

The contribution to the flexibility matrix of a set of mass normalized, undamped modes, is given by;

$$F = \tilde{\phi}\, \omega^{-2}\, \tilde{\phi}^{T} \tag{14}$$

where $\tilde{\phi}$ is the m x n modal matrix and ω is an (n x n) diagonal matrix listing the associated natural frequencies (m = # of sensors and n = # of identified modes). The modes that can be identified from the measured data are the complex modes of the damped system. Extraction of the undamped modes from the damped ones, for a system with an arbitrary viscous damping distribution, is a difficult problem for which an exact solution does not exist (and may not be feasible) when the sensors do not cover all the significant DOF[8]. If damping is classical (or is small), however, the undamped modes can be obtained as the displacement partition of the complex modes (after an appropriate rotation). In the plate considered here the damping is assumed classical and equal to 1% of critical in all the modes of the discretized system.

The results presented in this section are obtained by subjecting the plate to white noise excitation at sensors 4, 7 and 11 (see fig.2). The practical difficulties associated with attaining exact co-location of sensors and actuators are not considered. The response is computed using a transition matrix algorithm in a version that yields exact results if the variation of the excitation is linear between discretized points. A state space realization of the system is first obtained using the ERA-OKID algorithm[9] and the truncated flexibility matrix is extracted from the matrices of the realization using a procedure presented in Bernal [5]. Cases with and without noise are considered.

Results

The sampling rate for the excitation and the response is fixed at 100 Hz in all cases. Only damage case-a is considered and all the results are computed with mesh 1. The system has 166 elements and 268 unrestrained DOF. The exact changes in the frequencies resulting from the simulated damage are very small as can be seen from the plot in fig.4. One anticipates, therefore, that the damage may be difficult to locate under noisy conditions. The frequencies identified by the ERA-OKID algorithm for the case with 2% noise are listed for the undamaged state in Table 3 (the identified frequencies for the damaged case are not presented for brevity). Also depicted in the table are the % deviations from the values corresponding to the analytical model for frequencies up to approximately 80% of Nyquist. As can be seen, the accuracy of the modal identification is excellent.

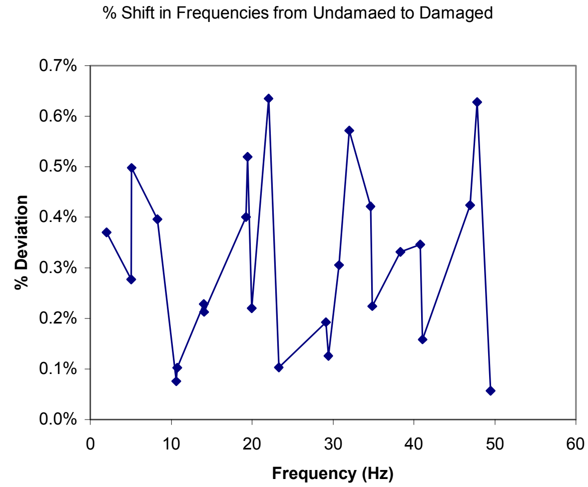


Fig.4 Percent change in modal frequencies between the undamaged and the damaged states.

Table 3. Identified frequencies and percent deviation from the analytical results for the undamaged case.

Identified Frequencies and % Error					
mode			mode		
1	2.0000	0.0000%	12	22.0340	0.0000%
2	5.0860	0.0118%	13	*	*
3	5.1210	0.0020%	14	29.1240	0.0206%
4	8.3040	0.0096%	15	29.4200	0.0170%
5	10.6310	0.0094%	16	30.7460	0.0033%
6	10.7580	0.0093%	17	32.0050	0.0062%
7	14.0210	0.0071%	18	34.6420	0.0087%
8	14.0770	0.0071%	19	34.8330	0.0086%
9	19.2230	0.0052%	20	38.3240	0.0130%
10	19.4280	0.0051%	21	40.7700	0.0123%
11	19.9640	0.0100%	22	41.0280	0.0317%
			23	46.9150	0.0234%

* mode not identified

In all the cases considered the order of the system was initially identified by the singular values of the Hankel matrix as 134 (67 modes). An examination of the modal co-linearity index[12] and the modal damping, however, was used to eliminate a large number of modes so only 23 modes were used in assembling the truncated flexibility. It is opportune to note that no attempt was made to identify modes in a higher frequency band by using signals sampled at a faster rate.

The nsi ratios are summarized in Table 4. Results are presented for three conditions, namely: a) no noise, b) 1% noise and c) 2% noise. The noise signals added to the output are scaled such that their RMS is the appropriate percent of that for the signal measured at sensor #5. In the noise added to the input the reference RMS is that for the actuator that is collocated with sensor #5.

Table 4. nsi values for the plate for the various noise levels considered.

0% Noise		1% Noise		2% Noise	
dlv 1		dlv 1 & 2		dlv 1 & 2	
element	nsi	element	nsi	element	nsi
***6**	**0.0642**	70	0.0642	80	0.0975
18	0.0749	***6**	**0.0749**	93	0.1110
32	0.0919	49	0.0919	120	0.2264
4	0.0920	3	0.0920	4	0.2369
64	0.1045	19	0.1045	47	0.2383
109	0.1052	47	0.1052	151	0.2414
34	0.1077	68	0.1077	148	0.2416
89	0.1112	84	0.1112	71	0.2430
151	0.1130	30	0.1130	123	0.2457
76	0.1320	45	0.1320	76	0.2540

* damaged element

As can be seen from the table, the damaged element is correctly identified in the no-noise and 1% noise cases but not in the case where the noise is 2%.

CONCLUSIONS

The paper presents a review of a recently introduced approach to localize linear damage in structural systems and examines its performance in the case of a simply supported square plate. The technique operates with the null space of the change in flexibility and locates the damage by inspection of the stress fields induced when the vectors in the null space are treated as loads at the sensor points. Examination of the conditions needed to ensure that the change in flexibility has a null space demonstrates that they cannot be strictly satisfied in continuous systems. In particular, a necessary condition is that the number of sensors be larger than the number of independent deformation modes of the damaged region and this number is ∞ in a continuous system.

Notwithstanding, an examination of the problem suggests that the DLV technique should lead to useful results if the number of deformation modes associated with a reasonably accurate discretization of a region that contains the damage is less than the number of sensors. The numerical results presented in the paper, although limited and exploratory in nature, support the previous contention.

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