A Flexibility Based Approach for the Localization and Quantification of Damage: Application in a Benchmark Structure

Dionisio Bernal and Burcu Gunes

Department of Civil and Environmental Engineering, 427 Snell Engineering Center, Northeastern University, Boston MA 02115, U.S.A

ABSTRACT

A flexibility based damage identification strategy is described and the results of its application to the analytical phase of the benchmark problem developed by the IASC-ASCE SHM Task Group are briefly summarized. The strategy contains localization and quantification modules that operate in cascade. The localization maps changes in measured flexibility to responsible elements in the structure using stress fields computed from load vectors defined by the null space of the flexibility change. The damage quantification is formulated as a quasi-linear optimization problem where the free parameters are those identified in the localization stage. The approach proved accurate with respect to the localization of the damage in all the cases considered. The accuracy of the quantification, however, is found to degrade notably in the presence of modeling error.

INTRODUCTION

Identification of damage from the analysis of vibration signals has received significant attention in the civil, mechanical and aerospace fields. The problem most commonly considered is that where data is recorded at two different times and it is of interest to determine if the structure suffered damage in the time interval between the two observations. The behavior of the system during the observation periods is typically assumed linear and the damage is identified as changes in system parameters. A solution, in principle, can be obtained by using the data to optimize a model of the structure in the two states and inspecting the differences. In practice, however, a solution is difficult because: a) the real structure is always more complex than the mathematical model selected b) the measured data is limited and imprecise and c) the number of system parameters is large and the inverse problem posed proves ill-conditioned.

Many techniques that try to circumvent or minimize the difficulties listed previously have been proposed [1]. Examination of the literature reveals, however, that the assumptions used to establish the various approaches vary widely and it is unclear what is the true capability of the current state of the art in damage detection of civil engineering structures. In an attempt to address this situation the dynamics committee of the ASCE formed a Task group on Structural Health Monitoring in 1999. The fundamental objective was to explore the damage identification problem with techniques selected by the participants but with the focus placed on a common structure and a set of common clearly defined assumptions. The details of the first selected benchmark structure, the damage scenarios, and a more elaborate discussion on the background and objectives of the ASCE SHM task group can be found in [2]. This paper discusses the techniques used by the authors in the phase 1 of the analytical work and presents a brief summary of the results obtained.

DAMAGE IDENTIFICATION STRATEGY

The sequence implemented has the following components:

1) Computation of a state-space realization from the measured signals.
2) Extraction of flexibility matrices from the matrices of the realization (to within a scalar multiplier when the input is stochastic).
3) Computation of the change in flexibility from the undamaged to the damaged state (or a matrix that differs from it by a scalar multiplier).
4) Reduction of the subset of potentially damaged elements by examination of the change in flexibility.
5) Quantification of the damage using the identified damaged flexibility.

The theme of the previous steps is the uncoupling of uncertainties in inertial and damping properties from the search for stiffness related damage. Since the originally recorded signals are reduced to flexibility matrices it is evident that the approach involves a substantial compression of the data. Whether or not the compression discards useful information (for the identification of stiffness related damage) depends on the position and number of sensors, on the number and frequency band of the identified modes, and on the spatial distribution of the damage. A detailed discussion, however, is not essential to the objectives of this paper and is omitted for brevity.

System Realization

For the deterministic case the state-space model in discrete time can be expressed as;

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}\,\boldsymbol{x}_k + \boldsymbol{B}\,\boldsymbol{u}_k \tag{1a}$$

$$\boldsymbol{y}_k = \boldsymbol{C}\,\boldsymbol{x}_k + \boldsymbol{D}\,\boldsymbol{u}_k \tag{1b}$$

where $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ and $\boldsymbol{D}$ are the matrices of the realization and $\boldsymbol{x}_k$ and $\boldsymbol{y}_k$ are the state and output vectors at step k. The Eigensystem Realization Algorithm with a Kalman

Observer [ERA-OKID] was used in the numerical analysis [3]. The selection of a finite system order results in some compression of the data.

In the unknown input case the output data is assumed to derive from a realization of the stochastic system;

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}\,\boldsymbol{x}_k + \boldsymbol{w}_k \tag{2a}$$

$$\boldsymbol{y}_k = \boldsymbol{C}\,\boldsymbol{x}_k + \boldsymbol{v}_k \tag{2b}$$

where $\boldsymbol{w_k}$ and $\boldsymbol{v_k}$ are white noise vectors. The objective of the analysis is to establish, exclusively from output measurements, the matrices $\boldsymbol{A}$ and $\boldsymbol{C}$ and the covariance of the process that generates the $\boldsymbol{w_k}$ and $\boldsymbol{v_k}$ sequences, namely;

$$E\left[\begin{pmatrix} \boldsymbol{w}_p \\ \boldsymbol{v}_p \end{pmatrix} \begin{pmatrix} \boldsymbol{w}_q^T & \boldsymbol{v}_q^T \end{pmatrix}\right] = \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{S} \\ \boldsymbol{S}^T & \boldsymbol{R} \end{bmatrix} \ddot{\boldsymbol{a}}_{pq} \tag{3}$$

The analysis assumes that the state is a zero mean stationary random process and that the disturbance and output noise vectors are not correlated with the state. The algorithm selected in this study, Sub-ID, belongs to the general class of subspace algorithms where the matrices of the realization are retrieved as subspaces of projected data matrices [4].

Extraction of Flexibility Matrices

The second level of data compression is realized by focusing on the behavior of the identified system at $\omega = 0$.

<u>Deterministic Input</u>

Taking a Fourier transform of eq.1 the realization can be expressed in input-output form, namely;

$$\boldsymbol{y}(\omega) = [\boldsymbol{C}\,[\boldsymbol{I}\cdot i\omega - \boldsymbol{A}]^{-1}\,\boldsymbol{B} + \boldsymbol{D}]\,\boldsymbol{u}(\omega) \tag{4}$$

It is convenient, for generality, to express the Fourier transform of the output vector in terms of the transform of the displacement vector, y_D;

$$\boldsymbol{y}(\omega) = (i\omega)^p\ \boldsymbol{y}_D(\omega) \tag{5}$$

where p = 0, 1 or 2 for displacement, velocity or acceleration sensing, respectively. Substituting eq.6 into eq.5 one gets;

$$\boldsymbol{y}_D(\grave{u}) = \frac{1}{(i\grave{u})^p}[\boldsymbol{C}\,[\boldsymbol{I}\cdot i\grave{u} - \boldsymbol{A}]^{-1}\,\boldsymbol{B} + \boldsymbol{D}]\,\boldsymbol{u}(\grave{u}) \tag{6}$$

Taking ω = 0 in eq.7 gives a partition of the flexibility matrix. After appropriate differentiation to eliminate indeterminacy one gets;

$$\boldsymbol{y}_{D(0)} = \boldsymbol{F}_{m,r}\, \boldsymbol{u}_{(0)} \tag{7}$$

where $\boldsymbol{F}_{m,r}$, the partition of the flexibility matrix associated with the *m* output sensors (rows) and *r* inputs (columns), is given by;

$$\boldsymbol{F_{m,r}} = -\boldsymbol{C}\, \boldsymbol{A}^{-(p+1)} \boldsymbol{B} \tag{8}$$

Note that eq.9 is general and thus applies equally well to cases with classical or non-classical damping. When all the inputs and outputs are co-located the equation provides the flexibility matrix directly. This condition, however, is rarely encountered in practice and further processing is required to extract all the available information. Indeed, the flexibility matrix can be obtained for all the input and output coordinates provided two conditions are satisfied: 1) there is at least one co-located pair of input-output sensors and 2) the damping distribution is classical. The reasons for these restrictions are noted in the sequel. Writing the partition of the flexibility matrix in eq.4 in terms of the undamped mass normalized modes and frequencies $\boldsymbol{\psi}$ and $\boldsymbol{\omega}$ one has;

$$\boldsymbol{F_{m,r}} = \boldsymbol{\psi}_m \ \text{ũ}^{-2} \ \boldsymbol{\psi}_r^T \tag{9}$$

where we note that the coordinates associated with co-located sensors are both in $\boldsymbol{\psi}_m$ and $\boldsymbol{\psi}_r$. Eqs. 9 and 10 can be written in series form and equated, one gets;

$$\sum_{i=1}^{n} \boldsymbol{\psi}_{m,i}\, \boldsymbol{\psi}_{r,i}^T \text{ũ}_i^{-2} = -2\Re \sum_{i=1}^{n} \boldsymbol{C}\text{ö} \ \overline{\text{ö}}^{\hat{o}} \boldsymbol{\Lambda}_i^{-(p+1)} \boldsymbol{B} \tag{10}$$

where *2n* is the system order and the notation of the right side comes from the spectral decomposition of *A*, namely;

$$\mathbf{A} = \text{ö} \quad \boldsymbol{\Lambda}\overline{\text{ö}} \tag{11}$$

where

$$\overline{\text{ö}} = \text{ö}^{-1} \tag{12}$$

To compute the mass normalized modes using eq.10 two conditions need to be satisfied: 1) the equality must hold term by term and 2) there must be at least one column (on the left side) where the number of unknows is *m* (and not *m*+1). The first requirement imposes a restriction to classical damping because it is only in this case that the displacement partition of $\boldsymbol{\Phi}_i$ is related to $\boldsymbol{\psi}_m$ by a complex constant. The equivalence of the second condition to the need for a co-located sensor is evident. Assuming the required conditions are satisfied the flexibility matrix at all the input and output coordinates can be computed as;

$$\boldsymbol{F} \approx \boldsymbol{\psi}\, \text{ũ}^{-2} \boldsymbol{\psi}^T \tag{13}$$

where $\boldsymbol{\psi} = [\boldsymbol{\psi}_m \ \boldsymbol{\psi}_r]^T$ and ≈ is introduced to emphasize that the modal space may be truncated. The term by term equality in eq.10 is only strictly valid for classical damping. The equality, however, is typically assumed to hold in general because modal complexity deriving from damping is small in lightly damped systems and in systems with well-separated frequencies [5].

Stochastic Input

In output-only systems flexibility matrices can not be assembled because the information to obtain mass normalized modes is not contained in the measured data. The damage localization strategy previously outlined can still be used, however, if a matrix that differs by a scalar multiplier from the true change in flexibility can be computed. Techniques to arrive at these matrices are presented in another paper contained in these proceedings [6].

Damage Localization

A general approach to map flexibility changes to spatial localization of damage, designated as the Damage Locating Vector (DLV) approach, has been recently developed. The basic features of the technique are described next, a more detailed discussion of the theoretical background as well as discussion on robustness and other issues may be found in [7].

The basic idea in the DLV approach is that the vectors that span the null-space of the change in flexibility from the undamaged to the damaged states, when treated as loads on the system, lead to stress fields that are zero over the damaged elements. Depending on the number and location of the sensors the intersection of the null stress regions identified by the DLVs may or may not contain elements that are not damaged in addition to the damaged ones. Elements that are undamaged but which cannot be discriminated from the damaged ones by changes in flexibility (for a given set of sensors) are inseparable. The steps of the DLV localization can be summarized as follows:

1. Compute the change in flexibility as;

$$\boldsymbol{DF} = \boldsymbol{F_U} - \boldsymbol{F_D} \tag{14}$$

2. Obtain a singular value decomposition of $\boldsymbol{DF}$, namely;

$$\boldsymbol{DF} = \boldsymbol{U}\begin{bmatrix} \boldsymbol{s}_1 & 0 \\ 0 & \boldsymbol{s}_2 \end{bmatrix}\boldsymbol{V}^T \tag{15}$$

where s_2 are 'small' singular values (the conditions for which the set s_2 is empty are discussed in [7]). For ideal conditions the s_2 values are zero and the DLV vectors are simply the columns of $\boldsymbol{V}$ associated with the null space. For the noisy conditions that prevail in practice, however, the values in s_2 are never equal to zero and a cutoff is needed to select the dimension of the effective null space. The steps to make the determination are presented next, the theoretical support can be found in [7] and is omitted for brevity.

1. Consider a vector in ***V*** say V_j
2. Compute the stresses in an undamaged model of the structure using V_j as loads.
3. Reduce the independent internal stresses in every element to a single characterizing stress, σ (defined in such a way that strain energy per unit characterizing dimension is proportional to σ^2). Designate the reciprocal of the maximum value of the characterizing stress as c_j.

4. Compute the *svn* index as;

$$svn_j = \sqrt{\frac{s_j\, c_j^2}{s_q\, c_q^2}} \tag{16}$$

where;

$$s_q\, c_q^2 = \max\,(s_j\, c_j^2) \qquad \text{for } j=1:\, m \tag{17}$$

5. The vector V_j can be treated as a DLV if;

$$svn_j \leq 0.20 \tag{18}$$

Once the DLV vectors have been identified the localization proper is carried out as follows:

6. Compute, for each DLV vector, the normalized stress index vector as;

$$nsi_i = \frac{\sigma_i}{\sigma_i\big|_{\max}}$$

(19)

7. Compute the vector of weighted stress indices, *WSI*, as;

$$WSI = \frac{\sum_{i=1}^{ndlv} \dfrac{nsi_i}{\overline{svn_i}}}{ndlv} \tag{20}$$

where,

$$\overline{svn_i} = \max\,(svn_i, 0.015) \tag{21}$$

and *ndlv* is the number of DLV vectors.

8. The potentially damaged elements are those having $WSI < 1$.

Quantification

The quantification implemented in phase 1 of the analytical work fits a model of the system to the damaged flexibility using the restricted set of free parameters identified in the localization stage. The metric selected to define the cost function is taken as the square of the deviations between the measured flexibility and the flexibility of the model. For this metric the problem can be cast as an 'almost' linear least square problem (strictly linear in statically determinate structures) that gives some insight into the issue of uniqueness.

To describe the approach, consider a discrete structure for which an arbitrary number of coordinates are defined. Assume, for simplicity, that the stiffness of the various elements is characterized by scalar quantities such as *EI*, *EA* or *GJ*. The reciprocals of these stiffness terms are element flexibilities, b_j, where the subscript identifies an element number. An arbitrary coefficient of the flexibility matrix can then be written as;

$$f_{i,j} = \boldsymbol{a}^T \boldsymbol{b} \tag{22}$$

where $\boldsymbol{a}$ is a vector of influence coefficients and $\boldsymbol{b}$ is the vector of element flexibilities. Placing all the coefficients of the upper triangular portion of the flexibility matrix in a vector one can write;

$$\boldsymbol{A}\boldsymbol{b} = \hat{\boldsymbol{f}} + \boldsymbol{\varepsilon} \tag{23}$$

where $\boldsymbol{A}$ contains a collection of influence vectors $\boldsymbol{a}^{\mathrm{T}}$, $\hat{\boldsymbol{f}}$ is the estimate of the damaged flexibility and $\boldsymbol{\varepsilon}$ is a vector of residuals which exists due to the fact that there is modeling error in the computation of $\boldsymbol{A}$ and truncation and approximation in the measured flexibility. It should be noted that since the parameters in $\boldsymbol{A}$ depend on the distribution of internal forces (due to unit loads) they are indirectly a function of the vector $\boldsymbol{b}$ so eq.26 is not linear (except in the statically determinate case). Ordering eq.26 so that the flexibility of the potentially damaged elements appears first and introducing the subscripts 1 and 2 to indicate partitions associated with potentially damaged and undamaged elements one gets;

$$\begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \end{bmatrix} \begin{Bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{Bmatrix} = \hat{\boldsymbol{f}} + \boldsymbol{\varepsilon} \tag{24}$$

or

$$\boldsymbol{A}_1 \boldsymbol{b}_1 = \Delta\hat{\boldsymbol{f}} + \varepsilon \tag{25}$$

where the first term on the right side of eq.28 is the total estimated flexibility minus the contribution that derives from the undamaged portion of the system. Defining the cost function as the squared length of the residual gives;

$$J(\boldsymbol{b_1}) = \frac{1}{2}\boldsymbol{\varepsilon}^T \boldsymbol{\varepsilon} = \boldsymbol{b}_1^T \boldsymbol{A}_1^T \boldsymbol{A}_1 \boldsymbol{b}_1 - \boldsymbol{b}_1^T \boldsymbol{A}_1^T \Delta\hat{\boldsymbol{f}} - \Delta\hat{\boldsymbol{f}}\, \boldsymbol{A}_1 \boldsymbol{b}_1 + \Delta\hat{\boldsymbol{f}}^T \Delta\hat{\boldsymbol{f}} \tag{26}$$

The minimum with respect to b_1 is obtained by setting the gradient to zero:

$$\frac{\partial J(\boldsymbol{b}_1)}{\partial \boldsymbol{b}_1} = \frac{1}{2}\left(2\boldsymbol{b}_1^T (\boldsymbol{A}_1^T \boldsymbol{A}_1) - 2\Delta\hat{\boldsymbol{f}}\, \boldsymbol{A}_1\right) = 0 \tag{27}$$

or;

$$\boldsymbol{b}_1 = (\boldsymbol{A}_1^T \boldsymbol{A}_1)^{-1} \boldsymbol{A}_1\, \Delta\hat{\boldsymbol{f}} + N(\boldsymbol{A}_1^T \boldsymbol{A}_1)\hat{\boldsymbol{a}} \tag{28}$$

where $\boldsymbol{\beta}$ is a vector of appropriate dimension and $\mathcal{N}(\cdot)$ indicates null space. Although mathematically arbitrary, the vectors $\boldsymbol{\beta}$ that lead to physically feasible solutions are limited by the fact that $\boldsymbol{b}_1$ is positive and bounded from below by the undamaged flexibility. Furthermore, note that any parameter associated with identically zero rows in the null space of $\boldsymbol{A}_1^{\mathrm{T}} \boldsymbol{A}_1$ can be computed independently of rank deficiency in $\boldsymbol{A}_1^{\mathrm{T}} \boldsymbol{A}_1$. We conclude by recalling that, with the exception of statically determinate systems, eq.31 needs to be solved using iterations because $\boldsymbol{A}$ is dependent on $\boldsymbol{b}$. Although a theoretical convergence analysis has not been attempted at the time of writing, no convergence difficulties were encountered in the cases examined.

SUMMARY OF THE RESULTS FOR THE BENCHMARK STRUCTURE

The simulation cases for the benchmark structure are summarized in Table I. The first 2 cases are one-dimensional analysis in the weak (y) direction with "ambient" excitation as the input. In case 2, modeling errors are introduced since the data is generated with a 120 DOF model. The level of complexity is increased in the subsequent cases. Case 3 replaces the ambient excitation at each floor with a shaker excitation on the roof. Cases 4 and 5 enhance the 3-D response by asymmetrical mass distribution on the top floor and further complicate the problem by asymmetrical damage patterns (damage patterns (3) and (4)). Tables II-IV display the results of modal identification, damage localization and quantification for Case 4. The results obtained for the other cases can be found in [8].

TABLE I. SIMULATION CASES FOR THE BENCHMARK STRUCTURE

	Case 1	Case 2	Case 3	Case 4	Case 5
Data Generation Model	12 DOF	120 DOF	12 DOF	120 DOF	120 DOF
Excitation	Ambient	Ambient	Shaker	Shaker	Shaker
ID Model	12 DOF	12 DOF	12 DOF	12 DOF	12 DOF
Input	Known Unknown	Known Unknown	Known Unknown	Unknown	Unknown
Damage Pattern*	(1), (2)	(1), (2)	(1), (2)	(1),(2),(3),(4)	(1),(2),(3),(4)

*(1) all braces in 1st story, (2) all braces in 1st and 3rd stories,
(3) 1 brace in 1st story, (4) 1 brace in 1st and 3rd stories

No difficulties were encountered in the modal identification. As expected, however, the accuracy proved higher in the known input cases.

TABLE II. IDENTIFIED MODAL PARAMETERS FOR CASE 4

No Damage		Damage (1)		Damage (2)		Damage (3)		Damage (4)	
ξ(%)	f (Hz)	ξ (%)	f (Hz)	ξ(%)	f (Hz)	ξ(%)	f(Hz)	ξ(%)	f(Hz)
1.23	9.29	1.14	6.14	1.18	5.70	1.19	8.76	1.10	8.83
1.02	11.68	1.03	9.80	1.26	9.39	1.06	11.68	0.97	11.52
0.96	25.24	0.95	21.28	1.05	14.82	1.39	15.84	1.73	15.66
1.27	31.60	0.95	28.57	0.84	24.71	1.06	24.39	1.05	24.37
0.89	38.27	0.94	36.96	1.23	36.02	1.27	31.61	0.89	30.83
1.22	48.07	1.03	37.96	0.92	40.55	0.88	37.81	0.95	37.73
0.99	59.86	0.95	46.87	1.10	46.45	0.95	43.63	0.96	42.87
1.21	66.88	1.23	47.59	0.85	53.62	0.88	47.68	0.88	47.56
1.11	83.33	1.01	59.67	1.05	63.48	1.21	48.08	1.00	48.14
		1.05	64.62	1.03	71.70	1.00	59.85	1.08	58.24
		1.10	82.94			1.20	66.55	0.97	66.60
						1.10	83.30	1.46	81.64

For all cases and damage patterns, the localization was performed successfully as shown in Table III. Characterizing stress is taken as the story shears and the *WSI* is computed for all the damage patterns. In all cases examined, the DLVs produced a set that is exclusively the damaged region.

TABLE III. DAMAGE LOCALIZATION AND THE WSI INDICES FOR CASE 4

	Damage (1)		Damage (2)			Damage (3)		Damage (4)	
Direction	Floor	WSI	Floor	WSI	Frame	Floor	WSI	Floor	WSI
x	1	0.14	1	0.07	East side (*y*-dir)	1	0.80	1	0.95
	2	4.90	2	11.47		2	8.35	2	3.27
	3	6.41	3	0.04		3	6.91	3	4.57
	4	9.04	4	11.82		4	7.46	4	5.26
y	1	0.21	1	0.01	South side (*x*-dir)	1	11.64	1	11.32
	2	5.07	2	13.89		2	26.35	2	11.10
	3	6.35	3	0.17		3	24.92	3	1.18
	4	8.65	4	11.85		4	21.11	4	8.10

When the damage is symmetric (Damage patterns (1) and (2)) it can be meaningfully characterized as percent change in the story stiffness for the complete system. When the damage introduces asymmetry such as damage patterns (3) and (4), however, it is necessary to quantify the damage at the level of the individual frames. The reference undamaged stiffnesses for each of the perimeter frames are specified as the analytical values associated with a triangular load distribution.

The error in the quantification of the percent damage for the extensive damage cases (patterns (1) and (2)) proved to be less than 4% for the data generated by the 12 DOF model and 9% for the data generated by the 120 DOF model. For damage patterns (3) and (4), however, the percent error in the quantification varies from 23-28%.

TABLE IV PERCENT REDUCTIONS IN IDENTIFIED STIFFNESSES FOR CASE 4

Floor	Damage (1)		Damage (2)		Damage (3)		Damage (4)	
	x-dir	*y-dir*	*x-dir*	*y-dir*	*y-dir (east)*	*x-dir (south)*	*y-dir (east)*	*x-dir (south)*
1	46.1	71.4	45.9	71.4	28.1	0.0	27.9	0.0
2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
3	0.0	0.0	47.1	0.0	0.0	0.0	0.0	27.7
4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

CONCLUSIONS

The paper describes a flexibility-based technique to locate and quantify linear damage and illustrates the performance in the case of the ASCE SHM task group benchmark structure. The approach operates by compressing the time histories into flexibility matrices synthesized at the sensor locations and uses these matrices as targets for identifying the location and severity of the damage.

A potentially important limitation in the strategy derives from the ability to assemble flexibility matrices from the measured data when only output signals are available. As the paper shows, however, most of the difficulties can be circumvented in the stochastic case if there are sufficient sensors because one can then synthesize matrices that differ from the flexibility by a single scalar. The scalar proves to be immaterial for localization purposes and is only of secondary importance in quantification since the reductions in stiffness can be computed in percentage without reference to specific values.

The analytical simulations of the benchmark problem have included most of the complications that are encountered in actual applications. The complexity of realistic field conditions is, of course, never fully captured by simulations so experimental validations of damage identification techniques are essential.

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